UTS

Stochastic Processes and Financial Mathematics

Lab/Tutorial 8

This lab is not assessed.

Question 1. Discrete-time Markov chain

Suppose the weather tomorrow depends on the weather yesterday and today. Specifically:

- if rain yesterday and rain today, then rain tomorrow with probability 0.4
- if rain yesterday and no rain today, then rain tomorrow with probability 0.2
- if no rain yesterday and rain today, then rain tomorrow with probability 0.3
- if no rain yesterday and no rain today, then rain tomorrow with probability 0.1

Let X_t, $t = \{0,1,2,\dots\}$, be a Markov chain taking states

$$X_t = \begin{cases} 1, & \text{rain yesterday, rain today (RR)} \\ 2, & \text{rain yesterday, no rain today (RN)} \\ 3, & \text{no rain yesterday, rain today (NR)} \\ 4, & \text{no rain yesterday, no rain today (NN)} \end{cases}.$$

(a) Write down the one-step transition probability matrix $\boldsymbol{P} \equiv \boldsymbol{P}(1) = [\text{Prob}(X_{t+1} = j | X_t = i)]_{i,j}$.

Hint 1. Note that RR $\longrightarrow$ RR ($1 \longrightarrow 1$) and RR $\longrightarrow$ RN ($1 \longrightarrow 2$) are the possible state changes for $X_t = 1$, with the other state changes from $X_t = 1$ having zero probability.

Hint 2. Work out the allowable state changes for $X_t = 2$, $X_t = 3$ and $X_t = 4$.

(b) Let the distribution of X_t be $\boldsymbol{p}(t) = [\text{Prob}(X_t = i)]_i$ and set

$$\boldsymbol{p}(0) = \begin{bmatrix} 0.10 \\ 0.15 \\ 0.20 \\ 0.55 \end{bmatrix},$$

Using R, compute $E[X_4]$, $E[X_{21}]$ and $E[X_{52}]$.

Hint 3. For matrix power, use R syntax `%^%` from `expm` package.

(c) Using R (or calculating by hand), find a stationary distribution π.

Hint 4. Use R function `qr.solve` or `eigen`.

(d) Is X_t ergodic?.

Question 2. Discrete-time Markov chain

Determine if the following homogenous discrete-time Markov chains are ergodic.

(a) One-step transition probability matrix

$$\boldsymbol{P} \equiv \boldsymbol{P}(1) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1/5 & 0 & 4/5 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

(b) One-step transition probability matrix

$$\boldsymbol{P} \equiv \boldsymbol{P}(1) = \begin{pmatrix} 1/6 & 1/2 & 0 & 0 & 1/3 \\ 1/4 & 1/4 & 0 & 0 & 1/2 \\ 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 6/7 & 0 & 1/7 \\ 1/10 & 3/10 & 0 & 0 & 3/5 \end{pmatrix}.$$

Question 3. Continuous-time Markov chain

Let X_t, t$\geq$ 0, be a homogenous continuous-time homogenous Markov chain taking states

$$X_t = \begin{cases} 1, & \text{A rating (low credit risk)} \\ 2, & \text{B rating (medium credit risk)} \\ 3, & \text{C rating (highcredit risk)} \\ 4, & \text{4, D rating (default)} \end{cases}$$

with intensity matrix

$$\boldsymbol{A} = \begin{bmatrix} -4 & 3 & 4/5 & 1/5 \\ 6/4 & -5 & 11/4 & 3/4 \\ 1 & 2 & -7 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and rates given per year.

(a) Find the transition probability matrix P^{jump} that applies when a jump occurs.

(b) Is X_t ergodic?

(c) Classify the states and identify two classes of X_t.

(d) Using R, in a single chart and including a legend, plot the probabilities of default

$$p_{i,4}(s) = \text{Prob}(X_{t+s} = 4 | X_t = x_i), i \in \{1,2,3\}, \qquad 0 \leq s \leq 10.$$

Hint 1. Use R function `lines` to add additional plots to chart and R function `legend` to add a legend.

(e) Using R, find the default probabilities

$$\text{Prob}(\tau \leq 12 | X_1 = 3)$$

and

$$\text{Prob}(\tau \leq 17 | X_3 = 2)$$

where τ is time of next default.

Question 4. Continuous-time Markov chain

Let a homogenous, continuous-time Markov chain have transition probability matrix (given jump occurs)

$$P^{\text{jump}} = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 3/4 & 1/4 & 0 \end{bmatrix}$$

with expected sojourn times $E[T_1] = 2$, $\text{var}(T_2) = 4$ and $E[T_3^2] = 1/8$.

(a) Find the generator matrix.

(b) Without using computational software, find a stationary distribution.

(c) Is the chain ergodic?