Numerical Methods 35006
Computer Lab 11: Finite Differences for ODEs and PDEs

For this Lab we will be using a combination of all the packages that we've used before in this subject:

```
import numpy as np
import matplotlib.pyplot as plt
import scipy.linalg as la
import scipy.sparse as sp
import scipy.sparse.linalg as sla
from mpl_toolkits.mplot3d import Axes3D
```

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1. a) Create a numpy array corresponding to a set of 100 points in the interval $0 \leq x \leq 4\pi$, then plot the point `f` which give the values of $f(x) = \sin x$ over this range.

   b) Using `sp.diags`, create a sparse matrix `D1` that performs a first derivative on `f` using central differences. Check this by plotting.

   c) Create a sparse matrix `D2` that performs a second derivative on `f`, and check this by plotting.

2. Using your matrix `D2` from Q1 together with the identity matrix creation function `sp.eye`, set up a sparse matrix equation to solve the two-point boundary value problem

$$-\frac{d^2u}{dx^2} + u = 1$$

   with the boundary conditions

$$u(0) = u(1) = 0 \ .$$

   Solve this using `sla.spsolve`. Plot your solution.

3. Use sparse matrices to solve the two-point boundary value problem

$$-\frac{d^2u}{dx^2} + 4\frac{du}{dx} + u = xe^{-x}$$

   on the domain $0 \leq x \leq 2$, and with the boundary conditions

$$u(0) = 0 \quad , \qquad u'(2) = -1 \ .$$

4. Use either `sla.eigs` or your own sparse eigenvalue solver from Lab 9 to compute at least one eigenfunction of the problem

$$-\frac{d^2u}{dx^2} = \lambda u$$

   with the boundary conditions

$$u(0) = u(1) = 0 \ .$$

   Plot your solution.

The next three questions are aimed at getting you to set up and solve Laplace's equation numerically.

5. a) Using your function definition and plotting from Lab 4, Q3, set up 2D arrays `X` and `Y` and plot the function

$$f(x, y) = (x-1)^2 + y^2$$

   using 10×10 grid points on the domain $-2 \leq x \leq 2$ and $-1 \leq y \leq 1$.

   b) Use `np.reshape` to turn the (N, N) matrices `X` and `Y` into np.arrays `Xv, Yv` with shape $(N^2, 1)$. Plot the values of `Xv` and `Yv` to check them.

6. a) Create a matrix `D2x` that performs the numerical partial derivative $\partial^2/\partial x^2$ and check this by plotting the derivative of your function f from the previous question.

   b) Create a matrix `D2y` that performs the numerical partial derivative $\partial^2/\partial y^2$.

7. Create a sparse matrix that numerically implements the Laplacian differential operator

$$\nabla^2 := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

   and use this to solve

$$\nabla^2 u = 0$$

   on the domain $-2 \leq x \leq 2$ and $-1 \leq y \leq 1$, with boundary conditions

$$u(-2, y) = u(2, y) = u(x, -1) = 0 \ , \quad u(x, 1) = 1 \ .$$

   Plot your solution.