# Class Test 2

St.Id: Name:

There are three questions in this test, total time 50 min.

Q1. Consider the following LP problem: 10 marks

$$\min z = -4x_1 - 5x_2$$
$$s.t. \quad 2x_1 + 3x_2 \le 6$$
$$3x_1 + x_2 \ge 3$$
$$x_1,\ x_2 \ge 0$$

Use one of Simplex methods in tabular form to solve the problem:

a) Specify the initial basis
b) Write the initial tableau in a canonical form
c) Solve the problem with the chosen Simplex method
d) State the optimal value of objective function and the optimal solution.

1. Using Dual Simplex method

$\min z = -4x_1 - 5x_2$

a) s.t $2x_1 + 3x_2 + s_1 = 6$

$3x_1 + x_2 - e_2 = 3 \rightarrow -3x_1 - x_2 + e_2 = -3$

$x_1\ x_2\ s_1\ e_2 \ge 0$

1

$X_{B_0} = (s_1, e_2)$ 1

canonical form

b)

| | $x_1$ | $x_2$ | $s_1$ | $e_2$ | RHS |
|---|---|---|---|---|---|
| z | 4 | 5 | 0 | 0 | 0 |
| $s_1$ | 2 | 3 | 1 | 0 | 6 |
| $e_2$ | -3 | -1 | 0 | 1 | -3 |

1

Use the next page to provide the solution:

Please provide your solution for Q1 here:

1

| | $x_1$ | $x_2$ | $s_1$ | $e_2$ | RHS |
|---|---|---|---|---|---|
| $z$ | 4 | 5 | 0 | 0 | 0 |
| $s_1$ | 2 | 3 | 1 | 0 | 6 |
| $e_2$ | $\boxed{-3}$ | -1 | 0 | 1 | -3 |

Ratio : $\left|\frac{4}{-3}\right| < \left|\frac{5}{-1}\right|$

$-3 < 0 \xrightarrow{e_2}$ is leaving 1

| | $x_1$ | $x_2$ | $s_1$ | $e_2$ | RHS |
|---|---|---|---|---|---|
| $z$ | 0 | $11/3$ | 0 | $4/3$ | -4 |
| $s_1$ | 0 | $\boxed{7/3}$ | 1 | $2/3$ | 4 |
| $x_1$ | 1 | $1/3$ | 0 | $-1/3$ | 1 |

$R_0' = R_0 - 4R_2'$

$R_1' = R_1 - 2R_2'$ 1

$R_2' = R_2 / (-3)$

Ratio : $4 \div 7/3 < 1 \div 1/3$

| | $x_1$ | $x_2$ | $s_1$ | $e_2$ | RHS |
|---|---|---|---|---|---|
| $z$ | 0 | 0 | $-\frac{11}{7}$ | $2/7$ | $-\frac{72}{7}$ |
| $x_2$ | 0 | 1 | $3/7$ | $\boxed{2/7}$ | $\frac{12}{7}$ |
| $x_1$ | 1 | 0 | $-1/7$ | $-\frac{3}{7} < 0$ | $\frac{3}{7}$ |

$R_0' = R_0 - \frac{11}{3} R_1'$

$R_1' = R_1 \times \frac{3}{7}$ 1

$R_2' = R_2 - \frac{1}{3} R_1'$

| | $x_1$ | $x_2$ | $s_1$ | $e_2$ | RHS |
|---|---|---|---|---|---|
| $z$ | 0 | -1 | -2 | 0 | -12 |
| $e_2$ | 0 | $7/2$ | $3/2$ | 1 | 6 |
| $x_1$ | 1 | $-3/2$ | $1/2$ | 0 | 3 |

$R_0' = R_0 - R_1$

$R_1' = R_1 \times \frac{2}{7}$ 1

$R_2' = R_2 + \frac{3}{7} R_1'$

The Tableau is optimal, as $\hat{c}_N \leq 0$ 2

$z^* = -12$

$x_1^* = 3$

$x_2^* = 0$

1

Q2 is on page 3

## 2. Big M method

$$\min z = -4x_1 - 5x_2$$

s.t. $2x_1 + 3x_2 \le 6$

$3x_1 + x_2 \ge 3$

$x_1,\ x_2 \ge 0$

Standard form:

$$\min z = -4x_1 - 5x_2 + Ma_2$$ 1

s.t. $2x_1 + 3x_2 + s_1 = 6$

$3x_1 + x_2 - e_2 + a_2 = 3$ 1

$x_1,\ x_2,\ s_1,\ e_2,\ a_2 \ge 0$

$x_{B_0} = (s_1, a_2)$; $z + 4x_1 + 5x_2 - Ma_2 = 0 \rightarrow$ NOT canonical $\rightarrow + MR_2$ 1 For for canonical form

1 $z + (3M+4)x_1 + (M+5)x_2 - Me_2 = 3M$

Ratio 1: $6/2 > 3/3$

| | $x_1$ | $x_2$ | $s_1$ | $e_2$ | $a_2$ | RHS |
|---|---|---|---|---|---|---|
| $z$ | $3M+4$ | $M+5$ | 0 | $-M$ | 0 | $3M$ |
| 1 $s_1$ | 2 | 3 | 1 | 0 | 0 | 6 |
| $a_2$ | $\boxed{3}$ | 1 | 0 | $-1$ | 1 | 3 |
| $z$ | 0 | $11/3$ | 0 | $4/3$ | $-M-4/3$ | $-4$ |
| 2 $s_1$ | 0 | $\boxed{7/3}$ | 1 | $2/3$ | $-2/3$ | 4 |
| $x_1$ | 1 | $1/3$ | 0 | $-1/3$ | $1/3$ | 1 |
| $z$ | 0 | 0 | $-\frac{11}{7}$ | $\frac{2}{7}$ | $-(M+4/3)-\frac{11}{7}$ | $-72/7$ |
| $x_2$ | 0 | 1 | $3/7$ | $2/7$ | $-2/7$ | $\frac{12}{7}$ |
| $x_1$ | 1 | 0 | $-1/7$ | $-3/7$ | $3/7$ | $3/7$ |
| $z$ | 0 | 0 | $-2$ | 0 | $-M-3$ | $-12$ |
| $e_2$ | 0 | 1 | $3/2$ | 1 | $-1$ | 6 |
| $x_1$ | 1 | $3/7$ | $1/2$ | 0 | 0 | 3 |

1 (first tableau)

Ratio 2: $4 \times \frac{3}{7} < 1 \times 3$

$R_0' = R_0 - (3M+4)R_2'$

1 $R_1' = R_1 - 2R_2$

$R_2' = R_2/3$

$R_0' = R_0 - \frac{11}{3}R_1'$

$R_1' = R_1 \times \frac{3}{7}$

1 $R_2' = R_2 - \frac{1}{3}R_1'$

$R_0' = R_0 - \frac{2}{7}R_1$

$R_1' = R_1 \times \frac{7}{2}$

1

$R_2' = R_2 + \frac{3}{7}R_1'$

The Tableau is optimal, as $\hat{c}_N \le 0$ 1

$z^* = -12$

$x_1^* = 3$

$x_2^* = 0$

1

3. Two-phase method

$$\min z = -4x_1 - 5x_2$$
$$\text{s.t.} \quad 2x_1 + 3x_2 \le 6$$
$$3x_1 + x_2 \ge 3$$
$$x_1,\ x_2 \ge 0$$

Standard form:

$$\min z = -4x_1 - 5x_2$$

s.t. $2x_1 + 3x_2 + S_1 = 6$ R1

$3x_1 + x_2 - e_2 + a_2 = 3$ R2(*). 1

$x_1\ x_2\ S_1\ e_2\ a_2 \ge 0$

Phase 1 Solve $\min w = a_2$
s.t. (*).

Initial basis $x_B = (S_1\ a_2)$ 1

$w - a_2 = 0$ → not in canonical form as $a_2$ is in basis.

\+
$3x_1 + x_2 - e_2 + a_2 = 3$

$w + 3x_1 + x_2 - e_2 = 3$

canonical 0.5

| | $x_1$ | $x_2$ | $S_1$ | $e_2$ | $a_2$ | RHS |
|---|---|---|---|---|---|---|
| $w$ | 3 | 1 | 0 | -1 | 0 | 3 |
| $S_1$ | 2 | 3 | 1 | 0 | 0 | 6 |
| $a_2$ | $\boxed{3}$ | 1 | 0 | -1 | 1 | 3 |
| $w$ | 0 | 0 | 0 | 0 | -1 | 0 |
| $S_1$ | 0 | 7/3 | 1 | 2/3 | -2/3 | 4 |
| $x_1$ | 1 | 1/3 | 0 | -1/3 | 1/3 | 1 |

$6 \div 2 > 3 \div 3$

$R_0' = R_0 - R_2$

$R_1' = R_1 - 2R_2'$

$R_2' = R_2/3$

1

$w^* = 0$ → proceed to Phase 2 1

Phase 2 $z + 4x_1 + 5x_2 = 0$

canonical 0.5

| | $x_1$ | $x_2$ | $s_1$ | $e_2$ | $a_2$ | |
|---|---|---|---|---|---|---|
| $z$ | 4 | 5 | 0 | 0 | | 0 |
| $s_1$ | 0 | $7/3$ | 1 | $2/3$ | $-2/3$ | 4 |
| $x_1$ | 1 | $1/3$ | 0 | $-1/3$ | $1/3$ | 1 |

– not canonical as $x_1$ is in the basis

$R_0' = R_0 - 4R_2$

| | | | | | |
|---|---|---|---|---|---|
| $z$ | 0 | $11/3$ | 0 | $4/3$ | $-4$ |
| $s_1$ | 0 | $\boxed{7/3}$ | 1 | $2/3$ | 4 |
| $x_1$ | 1 | $1/3$ | 0 | $-1/3$ | 1 |

1

| | | | | | |
|---|---|---|---|---|---|
| $z$ | 0 | 0 | $-\frac{11}{7}$ | $\frac{2}{7}$ | $-72/7$ |
| $x_2$ | 0 | 1 | $3/7$ | $2/7$ | $\frac{12}{7}$ |
| $x_1$ | 1 | 0 | $-1/7$ | $-3/7$ | $3/7$ |

$R_0' = R_0 - \frac{11}{3}R_1'$

$R_1' = R_1 \times \frac{3}{7}$

$R_2' = R_2 - \frac{1}{3}R_1'$

1

| | | | | | |
|---|---|---|---|---|---|
| $z$ | 0 | 0 | $-2$ | 0 | $-12$ |
| $e_2$ | 0 | 1 | $3/2$ | 1 | 6 |
| $x_1$ | 1 | $3/7$ | $1/2$ | 0 | 3 |

$R_0' = R_0 - \frac{2}{7}R_1$

$R_1' = R_1 \times \frac{7}{2}$

$R_2' = R_2 + \frac{3}{7}R_1'$

1

The Tableau is optimal, as $\hat{c}_N \le 0$ 1

$z^* = -12$

$x_1^* = 3$

$x_2^* = 0$

1

Q2. For the following LP problem **10 marks**

$$\min z = 2x_1 + 3x_2 - x_3$$

$$\text{s.t.} \quad x_1 + 2x_2 + x_3 \geq 2$$

$$x_1 - x_2 - x_3 = 1$$

$$x_1 \ urs, \ \ x_2 \leq 0, \ \ x_3 \geq 0$$

State the dual problem, simplify its statement if possible. Provide a brief explanation on how the dual problem has been constructed.

Please provide your solution below:

1. THIS min problem is not IN Normal form:

$\min \ c^T x$

$s.t \ \ Ax \geq b$

$\downarrow$

1 Assymetrical dual.

➢ Asymmetric dual:

apply the following rules:

| primal/dual constraint | | dual/primal variable |
|---|---|---|
| consistent with normal form | $\iff$ | variable $\geq 0$ |
| reversed with normal form | $\iff$ | variable $\leq 0$ |
| equality constraint | $\iff$ | variable urs |

$\min z = 2x_1 + 3x_2 - x_3$

s.t. $x_1 + 2x_2 + x_3 \geq 2 \longrightarrow$ consistent with normal form; $y_1 \geq 0$ 1

$x_1 - x_2 - x_3 = 1 \rightarrow$ "=" constraint $\rightarrow y_2$ - URS 1

$\underbrace{x_1 \ urs}_{}$, $x_2 \leq 0$, $x_3 \geq 0 \longrightarrow$ 3rd dual constraint is consistent with normal form 1

1 $\downarrow$ 1

1st dual constraint is "="

2nd dual constraint "reverse" from normal form

Use the next page to provide the solution:

Please provide your solution for Q2 here:

Dual: max $2y_1 + y_2$ 1

s.t. $y_1 + y_2 = 2$ 1

$2y_1 - y_2 \geq 3$ 1

$y_1 - y_2 \leq -1$ 1

$y_1 \geq 0$, $y_2$ - URS

OR if not using assymetrical dual

$$\min z = 2x_1 + 3x_2 - x_3$$

$$s.t. \quad x_1 + 2x_2 + x_3 \geq 2$$

$$x_1 - x_2 - x_3 = 1$$

$$x_1\ urs,\ x_2 \leq 0,\ x_3 \geq 0$$

1. Bring to normal form:

$x_1 = x_1' - x_2''$ 1 ; $x_2' = -x_2$ 1

$x_1 - x_2 - x_3 = 1 \rightarrow \begin{matrix} x_1 - x_2 - x_3 \geq 1 \\ x_1 - x_2 - x_3 \leq 1 \end{matrix} \rightarrow \left.\begin{matrix} x_1 - x_2 - x_3 \geq 1 \\ -x_1 + x_2 + x_3 \geq -1 \end{matrix}\right\}$ 1

Primal in normal form

$$\left.\begin{aligned} \min\ z &= 2x_1' - 2x_1'' - 3x_2' - x_3 \\ \text{s.t. } & x_1' - x_1'' - 2x_2' + x_3 \geq 2 \\ & x_1' - x_1'' + x_2' - x_3 \geq 1 \\ & -x_1' + x_1'' - x_2' + x_3 \geq -1 \\ & x_1'\ x_1''\ x_2'\ x_3 \geq 0 \end{aligned}\right\}$$ 1

Q3 is on page 5

Symmetrical dual

$$\max \quad 2y_1 + y_2 - y_3$$

$$\left.\begin{aligned} \text{s.t.} \quad & y_1 + y_2 - y_3 \le 2 \\ & -y_1 - y_2 + y_3 \le -2 \\ & -2y_1 + y_2 - y_3 \le -3 \\ & y_1 - y_2 + y_3 \le -1 \\ & y_1\ y_2\ y_3 \ge 0. \end{aligned}\right\} \quad 1$$

Simplified Dual: let $u = y_2 - y_3$ 1

Then

$$\max \quad 2y_1 + u$$

$$1 \quad \left\{\begin{aligned} & \left.\begin{aligned} y_1 + u &\le 2 \\ -y_1 - u &\le -2 \end{aligned}\right\} \to \begin{aligned} y_1 + u &\le 2 \\ y_1 + u &\ge 2 \end{aligned} \to y_1 + u = 2 \quad 1 \\ & -2y_1 + u \le -3 \\ & y_1 - u \le -1 \\ & y_1 \ge 0; \ u - \text{URS} \quad 1 \end{aligned}\right.$$

Finally, simplified dual:

$$\left.\begin{aligned} \max \quad & 2y_1 + u \\ \text{s.t.} \quad & y_1 + u = 2 \\ & 2y_1 - u \ge 3 \\ & y_1 - u \le -1 \\ & y_1 \ge 0, \ u - \text{urs}. \end{aligned}\right\} \quad 1$$

Q3. Prove that if $A$ is an $n \times n$ positive definite matrix, then all eigenvalues of $A$ are positive.

<u>Please provide your solution below:</u>

Let $\lambda_i$ be an eigenvalue of $A$, and

$x_i$ be the corresponding eigenvector:

1 $Ax_i = \lambda_i x_i$ ; and $x_i \neq 0$ 1

As $A$ is pos.-def $\rightarrow$ for any $x \neq 0$ $x^T A x > 0$,

then $x_i^T A x_i = x_i^T \cdot \lambda x_i = \lambda |x_i|^2 > 0 \rightarrow$

1 1 $\lambda > 0$ 1