Modeling with Excel exercises

Part one

Go through the formulations of these problems in the set-based form. Then solve the problems with Excel.

Winston Problem 1 page 55

Farmer Jones must determine how many acres of corn and wheat to plant this year. An acre of wheat yields 25 bushels of wheat and requires 10 hours of labor per week. An acre of corn yields 10 bushels of corn and requires 4 hours of labor per week. All wheat can be sold at \$4 a bushel, and all corn can be sold at \$3 a bushel. Seven acres of land and 40 hours per week of labor are available. Government regulations require that at least 30 bushels of corn be produced during the current year. Formulate an LP whose solution will tell Farmer Jones how to maximise the total revenue from wheat and corn.

1) Define the variables
2) minimise (or maximise) the objective function
3) subject to
4) list of constraints
5) domain of variables

Solution explanation:
The scalar based model of this problem (see tutorial 1):

Let x_1 = number of acres of corn planted, and x_2 = number of acres of wheat planted.

$$\text{maximise} \quad z = 30x_1 + 100x_2$$

subject to

$$x_1 + x_2 \leq 7$$
$$4x_1 + 10x_2 \leq 40$$
$$10x_1 \geq 30$$
$$x_1 \geq 0, \quad x_2 \geq 0$$

To write a set based model we need to group related objects together. The objects are related (or similar) if they are treated in a similar way in the objective function and/or constraints. In this problem wheat and corn can be grouped together as a set, which will be named as crop. Accordingly, wheat and corn are all members of the set crop. The member of a set is referred to by an index which is an integer. You can imagine that the set is actually an array of members and the index is the position of each member in the array. We assign index to members of the set crop according to

Crop	corn	wheat
Index	1	2

A set can have attributes. For crop, it has price, allocated area, yield and required labour hours as attributes. We need to introduce notation to represent these attributes

Attribute	Notation	Explanation
Price	p	sale price in dollar for each bushel of the crop
Allocation	x	area allocated in acre to plant the crop
Yield	y	yield of the crop in bushels for each acre
Labour	l	labour hours per week for each acre of the crop

To refer to the attributes of a member, we use the notation which is the combination of the attribute notation and the index of the member. For example,

Notation	Explanation
p_1	sale price in dollar for each bushel of corn
p_2	sale price in dollar for each bushel of wheat

The values of the attributes of a member can be organised into tables.

	Index	
Attributes	1	2
p	3	4
x	x_1	x_2
y	10	25
l	4	10

To avoid using raw data in the algebraic model directly, we further introduce the notation

Notation	Value	Explanation
d	7	available acres of land
h	40	available labour hours per week
m	30	minimal bushels of corn to produce

By replacing the constants in the scalar model with the notations we introduced, Let $C = \{1, 2\}$ be the set of crop. Denote the price, yield and labour hours of crop i, $i = 1, 2$ by p_i, y_i and l_i respectively. The available acres of land is d, the available labour hours per week is h, and the minimal bushels of corn to produce is m. The acres of land allocated to each crop, denoted by x_i, $i = 1, 2$ are the decision variables.

$$\text{maximise} \quad z = y_1p_1x_1 + y_2p_2x_2$$

subject to

$$x_1 + x_2 \leq d$$

$$l_1x_1 + l_2x_2 \leq h$$

$$y_1x_1 \geq m$$

$$x_1 \geq 0, \quad x_2 \geq 0$$

We can see immediately that some terms in the model are exactly the same except for the subscript. We can write the summation of these similar terms in a nicer way using the $\sum$ notation. For example, the objective function can be rewritten as

$$\text{maximise} \quad z = y_1p_1x_1 + y_2p_2x_2 = \sum_{i=1}^{2} y_ip_ix_i$$

We can also rewrite the constraints as

$$x_1 + x_2 \leq d \to \sum_{i=1}^{2} x_i \leq d$$

$$l_1x_1 + l_2x_2 \leq h \to \sum_{i=1}^{2} l_ix_i \leq h$$

The non-negativity constraints on the decision variables can also be grouped together using "for all"

$$x_i \geq 0, \quad \text{for all } i = 1, 2$$

Then the model can be extended to consider arbitrary crops by replacing $C = \{1, 2\}$ by $C = \{1, 2, \ldots, n\}$.

Let $C = \{1, 2, \ldots, n\}$ be the set of crops. Denote the price, yield and labour hours of crop i, $i = 1, 2, \ldots, n$ by p_i, y_i and l_i respectively. The available acres of land is d, the available labour hours per week is h, and the minimal bushels of corn to produce is m. The acres of land allocated to each crop, denoted by x_i, $i = 1, 2, \ldots, n$ are the decision variables.

$$\text{maximise} \quad z = \sum_{i=1}^{n} y_i p_i x_i$$

subject to

$$\sum_{i=1}^{n} x_i \leq d$$

$$\sum_{i=1}^{n} l_i x_i \leq h$$

$$y_1 x_1 \geq m$$

$$x_i \geq 0, \quad \text{for all } i = 1, 2, \ldots, n$$

Consider the extension of the problem and solve it with Excel: besides wheat and corn, Farmer Jones also wants to grow sugar beets. One acre of sugar beets will yield 50 tons of sugar beets and require 1 hour of labour per acre. Each ton of sugar beets can be sold for $1. An additional government regulation is introduced where 50 tons of wheat are required.

Winston Problem 1 page 92

You have decided to enter the candy business. You are considering producing two types of candies: Slugger Candy and Easy Out Candy, both of which consist solely of sugar, nuts, and chocolate. At present, you have in stock 100 oz of sugar, 20 oz of nuts, and 30 oz of chocolate. The mixture used to make Easy Out Candy must contain at least 20% nuts. The mixture used to make Slugger Candy must contain at least 10% nuts and 10% chocolate. Each ounce of Easy Out Candy can be sold for 25 cents , and each ounce of Slugger Candy for 20 cents. Formulate an LP that will enable you to maximise your revenue from candy sales.

Solution explanation:
The scalar based model of this problem (see tutorial 1):

Let x_1 = amount of sugar used to make Easy Out Candy;
x_2 = amount of nut used to make Easy Out Candy;
x_3 = amount of chocolate used to make Easy Out Candy;
y_1 = amount of sugar used to make Slugger Candy;
y_2 =amount of nut used to make Slugger Candy;
y_3 =amount of chocolate used to make Slugger Candy.

$$\text{maximise} \quad z = 25(x_1 + x_2 + x_3) + 20(y_1 + y_2 + y_3)$$

subject to

$$x_1 + y_1 \leq 100$$
$$x_2 + y_2 \leq 20$$
$$x_3 + y_3 \leq 30$$
$$0.2(x_1 + x_2 + x_3) \leq x_2$$
$$0.1(y_1 + y_2 + y_3) \leq y_2$$
$$0.1(y_1 + y_2 + y_3) \leq y_3$$
$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad y_1 \geq 0, \quad y_2 \geq 0, \quad y_3 \geq 0$$

By replacing the constants in the scalar model with the notations, the LP model becomes follow:

Let $R = \{1, 2\}$ be the set of ingredients.
Let x_1 = amount of sugar used to make Easy Out Candy;
x_2 = amount of nut used to make Easy Out Candy;
x_3 = amount of chocolate used to make Easy Out Candy;
y_1 = amount of sugar used to make Slugger Candy;
y_2 =amount of nut used to make Slugger Candy;
y_3 =amount of chocolate used to make Slugger Candy;
P_1 = the price of Easy Out Candy;
P_2 = the price of Slugger Candy;
A_1 = the availability of sugar;
A_2 = the availability of nuts
A_3 = the availability of chocolate;
e_1 = the requirement of sugar for Easy Out Candy;
e_2 = the requirement of nuts for Easy Out Candy;
e_3 = the requirement of chocolate for Easy Out Candy;
s_1 = the requirement of sugar for Slugger Candy;
s_2 = the requirement of nuts for Slugger Candy;
s_3 = the requirement of chocolate for Slugger Candy.

$$\text{maximise} \quad z = P_1(x_1 + x_2 + x_3) + P_2(y_1 + y_2 + y_3)$$

subject to

$$x_1 + y_1 \leq A_1$$

$$x_2 + y_2 \leq A_2$$

$$x_3 + y_3 \leq A_3$$

$$e_2(x_1 + x_2 + x_3) \leq x_2$$

$$s_2(y_1 + y_2 + y_3) \leq y_2$$

$$s_3(y_1 + y_2 + y_3) \leq y_3$$

$$x_1 \geq 0, x_2 \geq, x_3 \geq 0, y_1 \geq 0, y_2 \geq 0, y_3 \geq 0$$

Now, the candy business wants to add some new ingredients into both Easy out Candy and Slugger Candy. To make our model more adaptive, we extend this model into a set based model for arbitrary ingredients. The set based model is

Let $R = \{1, 2, \ldots, n\}$ be the set of ingredients.
x_i=the amount of ingredient $i \in R$ added to make Easy Out Candy;
y_i=the amount of ingredient $i \in R$ added to make Slugger Candy;
P_1 = the price of Easy Out Candy;
P_2 = the price of Slugger Candy;
A_i = the availability of ingredient $i \in R$;
e_i = the requirement of ingredient $i \in R$ for Easy Out Candy;
s_i = the requirement of ingredient $i \in R$ for Slugger Candy;

$$\text{maximise} \quad z = P_1 \sum_{i=1}^{n} x_i + P_2 \sum_{i=1}^{n} y_i$$

subject to

$$x_i + y_i \leq A_i, \quad \text{for all } i = 1, 2, \ldots, n.$$

$$e_i \sum_{j=1}^{n} x_j \leq x_i, \quad \text{for all } i = 1, \ldots, n.$$

$$s_i \sum_{j=1}^{n} y_j \leq y_i, \quad \text{for all } i = 1, \ldots, n.$$

$$x_i \geq 0, \quad \text{for all } i = 1, 2, \ldots, n.$$

$$y_i \geq 0, \quad \text{for all } i = 1, 2, \ldots, n.$$

Part 2: More complex problems - Double Subscripting

Consider the following example (taken from the online LINGO user manual) …

For our example, suppose that the Wireless Widget (WW) Company has six warehouses supplying eight vendors with their widgets. Each warehouse has a supply of widgets that cannot be exceeded, and each vendor has a demand for widgets that must be satisfied. WW wants to determine how many widgets to ship from each warehouse to each vendor so as to minimize the total shipping cost. This is a classic optimization problem referred to as the *transportation* problem.

Widget Capacity Data:

Warehouse	Widgets On Hand
1	60
2	55
3	51
4	43
5	41
6	52

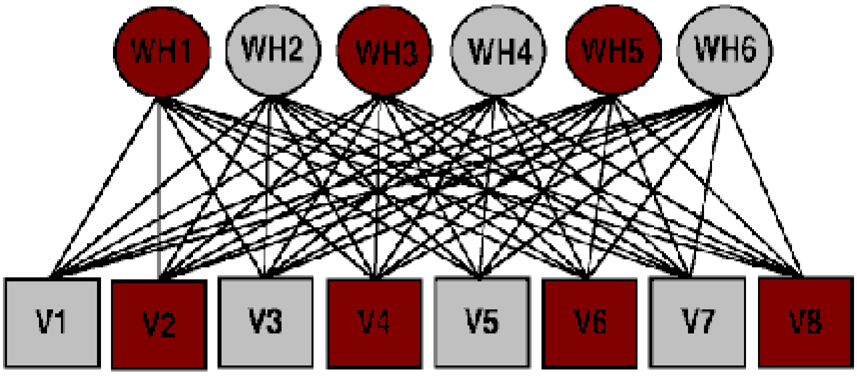
Vendor Widget Demand:

Vendor	Widget Demand
1	35
2	37
3	22
4	32
5	41
6	32
7	43
8	38

Shipping Cost per Widget ($):

	V1	V2	V3	V4	V5	V6	V7	V8
WH1	6	2	6	7	4	2	5	9
WH2	4	9	5	3	8	5	8	2
WH3	5	2	1	9	7	4	3	3
WH4	7	6	7	3	9	2	7	1
WH5	2	3	9	5	7	2	6	5
WH6	5	5	2	2	8	1	4	3

The following diagram illustrates the problem:



Wireless Widget's Shipping Network

Since each warehouse can ship to each vendor, there are a total of 48 possible shipping paths, or arcs. We will need a variable for each of these arcs to represent the amount shipped on the arc.

Let

Xij = the number of widgets sent from warehouse j, j = 1, …, 6, to vendor i, i = 1, …, 8;

Cj = the number of widgets on hand at warehouse j, j = 1, …, 6;

Di = the widget demand at vendor i, i = 1, …, 8;

Sij = the unit shipping cost from warehouse j, j = 1, …, 6, to vendor i, i = 1, …, 8;

$$\min\ z = \sum_{i=1}^{8}\sum_{j=1}^{6} SijXij$$

Subject to

$$\sum_{j=1}^{6} Xij \geq Di, for\ all\ i = 1, \dots, 8$$

$$\sum_{i=1}^{8} Xij \leq Cj, for\ all\ j = 1, \dots, 6$$

$$Xij \geq 0, for\ all\ i = 1, \dots, 8, j = 1, \dots, 6$$

The use of sets makes more sense when dealing with larger scale problems.

In a transportation problem, we're shipping from warehouses to vendors (or Factories to Warehouses), and each of these things can be considered to be a set. Each decision variable depends on both the warehouse and the vendor and is hence double subscripted. We need to define a set, {Warehouse, Vendor}, which in turn depends on the sets {Warehouse} and {Vendors}. Solve the problem with Excel

Blending problem

A one-kilogram pack of dog food must contain protein (at least 31%), carbohydrate (at least 38%), and fat (between 13% and 15%, that is greater than or equal to 13% and less than or equal to 15%). Three foods (food1, food2 and food3) are to be blended together in various proportions to produce a least-cost pack of dog food satisfying these requirements. The information for one kilogram of each food is given in the table below.

	protein (*grams*)	*carbohydrate* (*grams*)	*fat* (*grams*)	*price* (*dollars*)
food1	300	500	90	3
food2	450	300	160	1
food3	200	400	200	2

Let x_i be the amount (in kilograms) of food i to be used in the production of one kilogram of the final product (dog food).

$$\text{minimise} \quad z = 3x_1 + x_2 + 2x_3$$

subject to

$300x_1 + 450x_2 + 200x_3 \geq 310$ $500x_1 +$ 31% protein 38%
$300x_2 + 400x_3 \geq 380$ $90x_1 + 160x_2 + 200x_3$ carbohydrate min 13% fat
≥ 130 $90x_1 + 160x_2 + 200x_3 \leq 150$ max 15% fat

$$x_1 + x_2 + x_3 = 1$$

$$x_1 \geq 0, \; x_2 \geq 0, \qquad x_3 \geq 0$$

General form of blending problem

Assume we have m ingredients and n foods. Let a_{ij} be the grams of ingredient j in one Kilo of food i, c_i be the price of food i per Kilo. Each ingredient is only allowed within the range of $[l_j, u_j]$ in the final food.

minimise $z = 3x_1 + x_2 + 2x_3$

subject to

$300x_1 + 450x_2 + 200x_3 \geq 310$

$500x_1 + 300x_2 + 400x_3 \geq 380$

$90x_1 + 160x_2 + 200x_3 \geq 130$

$90x_1 + 160x_2 + 200x_3 \leq 150$

$x_1 + x_2 + x_3 = 1$

$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_3 \geq 0$

minimise $z = \sum_{i=1}^{n} c_i x_i$

subject to

$$\sum_{i=1}^{n} a_{ij} x_i \geq l_j, \qquad j = 1, \cdots, m$$

$$\sum_{i=1}^{n} a_{ij} x_i \leq u_j, \qquad j = 1, \cdots, m$$

$$\sum_{i=1}^{n} x_i = 1$$

$$x_i \geq 0, \qquad i = 1, \cdots, n$$