

Lecture Notes – Part 11

Integer Programming (II)

3 Combinatorial Optimisation Problems (cont.)

3.3 Travelling Salesman Problem (TSP)

The problem can simply be stated as follows. If a travelling salesman, who starts from the home city, wishes to visit exactly once each of a list of m cities, where the cost of travelling from city i to city j is c_{ij}^* , and then return to his home city, what is the least costly route the travelling salesman can take?

A binary IP for the TSP might be formulated as below. Firstly, set up the cost matrix $\mathbf{C}$ by

$$c_{ij} = \begin{cases} c_{ij}^*, & \text{if } i \neq j; \\ M, & \text{otherwise,} \end{cases}$$

where M is an extremely large positive number.

For each $i = 1, \dots, m$ and each $j = 1, \dots, m$, let

$$x_{ij} = \begin{cases} 1, & \text{if the salesman leaves city } i \text{ and travel next to city } j; \\ 0, & \text{otherwise.} \end{cases}$$

Then the binary IP could be shown as follows

$$\begin{aligned}
\min \quad & z = \sum_{i=1}^m \sum_{j=1}^m c_{ij} x_{ij} \\
s.t. \quad & \sum_{j=1}^m x_{ij} = 1, \quad i = 1, \dots, m, \\
& \sum_{i=1}^m x_{ij} = 1, \quad j = 1, \dots, m, \\
& x_{ij} = 0 \text{ or } 1.
\end{aligned}$$

The first m constraints ensure that the salesman leaves each city exactly one time. The last m ensure that the salesman visits each city just one time.

You may have noticed that it is exactly an IP model of the assignment problem. However, utilising the solution approach like Hungarian method alone for the above binary IP could result in separate cycles such that a single route cannot be generated. Actually, the above IP formulation for TSP hasn't yet been finished. To prevent separate cycles in TSP, we need

- add some additional constraints to the above binary IP, or
- use the B&B method which we are going to demonstrate.

Example 5. Noah Syndergaard lives in Gary, Indiana. He owns insurance agencies in Gary, Fort Wayne, Evansville, Terre Haute, and South Bend. Each December, he visits each of his insurance agencies. The distance between each agency (in miles) is:

	City 1 (GA)	City 2 (FW)	City 3 (EV)	City 4 (TH)	City 5 (SB)
City 1 (GA)	0	132	217	164	58
City 2 (FW)	132	0	290	201	79
City 3 (EV)	217	290	0	113	303
City 4 (TH)	164	201	113	0	196
City 5 (SB)	58	79	303	196	0

What order of visiting his agencies will minimise the total distance travelled?

Solution. We apply the B&B method to solve this TSP.

Take a large positive number M , say $M = 10000$. The cost matrix of the problem is

<u>SubP1</u>	City 1	City 2	City 3	City 4	City 5
City 1	M	132	217	164	58
City 2	132	M	290	201	79
City 3	217	290	M	113	303
City 4	164	201	113	M	196
City 5	58	79	303	196	M

Subproblem 1. For each $i = 1, \dots, 5$ and each $j = 1, \dots, 5$, let

$$x_{ij} = \begin{cases} 1, & \text{if the Noah leaves city } i \text{ and then travel to city } j; \\ 0, & \text{otherwise.} \end{cases}$$

The assignment-problem binary IP is

$$\begin{aligned}
\min \quad & z = \sum_{i=1}^5 \sum_{j=1}^5 c_{ij} x_{ij} \\
s.t. \quad & \sum_{j=1}^5 x_{ij} = 1, \quad i = 1, \dots, 5, \\
& \sum_{i=1}^5 x_{ij} = 1, \quad j = 1, \dots, 5, \\
& x_{ij} = 0 \text{ or } 1.
\end{aligned}$$

Applying the Hungarian method to the cost matrix of the above binary IP model, which is an assignment problem, we obtain an optimal solution $x_{15} = x_{52} = x_{21} = x_{34} = x_{43} = 1$ with $z_{\min} = 495$. This solution contains two subtours (1-5-2-1) and (3-4-3) and cannot be an optimal solution to the considered TSP. We will eliminate these subtours by branching Subproblem 1. We choose to exclude the subtour (3-4-3). This can be done by setting either $x_{34} = 0$ or $x_{43} = 0$. In other words, we branch on Subproblem 1 by adding each of these conditions. This yields following two subproblems:

- Subproblem 2 = Subproblem 1 + ($x_{34} = 0$ by setting $c_{34} = M$), and
- Subproblem 3 = Subproblem 1 + ($x_{43} = 0$ by setting $c_{43} = M$).

We now arbitrarily choose Subproblem 2 to solve. The table “SubP2” is the cost matrix for Subproblem 2.

<u>SubP2</u>	City 1	City 2	City 3	City 4	City 5
City 1	M	132	217	164	58
City 2	132	M	290	201	79
City 3	217	290	M	M	303
City 4	164	201	113	M	196
City 5	58	79	303	196	M

Applying the Hungarian method to this cost matrix, we obtain an optimal solution $x_{14} = x_{43} = x_{31} = x_{25} = x_{52} = 1$ with $z_{\min} = 652$. This solution contains two subtours (1-4-3-1) and (2-5-2) and again cannot be an optimal solution. We will eliminate these subtours by branching Subproblem 2. We choose to exclude the subtour (2-5-2). This can be done by setting either $x_{25} = 0$ or $x_{52} = 0$. In other words, we branch on Subproblem 2 by adding each of these conditions. This yields following two subproblems:

- Subproblem 4 = Subproblem 2 + ($x_{25} = 0$ by setting $c_{25} = M$), and
- Subproblem 5 = Subproblem 2 + ($x_{52} = 0$ by setting $c_{52} = M$).

We now arbitrarily choose Subproblem 4 to solve. The table “SubP4” below is the cost matrix for Subproblem 4.

<u>SubP4</u>	City 1	City 2	City 3	City 4	City 5
City 1	M	132	217	164	58
City 2	132	M	290	201	M
City 3	217	290	M	M	303
City 4	164	201	113	M	196
City 5	58	79	303	196	M

Applying the Hungarian method to this cost matrix yields an optimal solution $x_{15} = x_{52} = x_{24} = x_{43} = x_{31} = 1$ with $z_{\min} = 668$. Now this solution contains no subtours and yields a single route (1-5-2-4-3-1). Thus, Subproblem 4 produces a candidate for optimal solution and an upper bound: Any subproblem that has its objective value no less than $UB = 668$ will be eliminated from consideration.

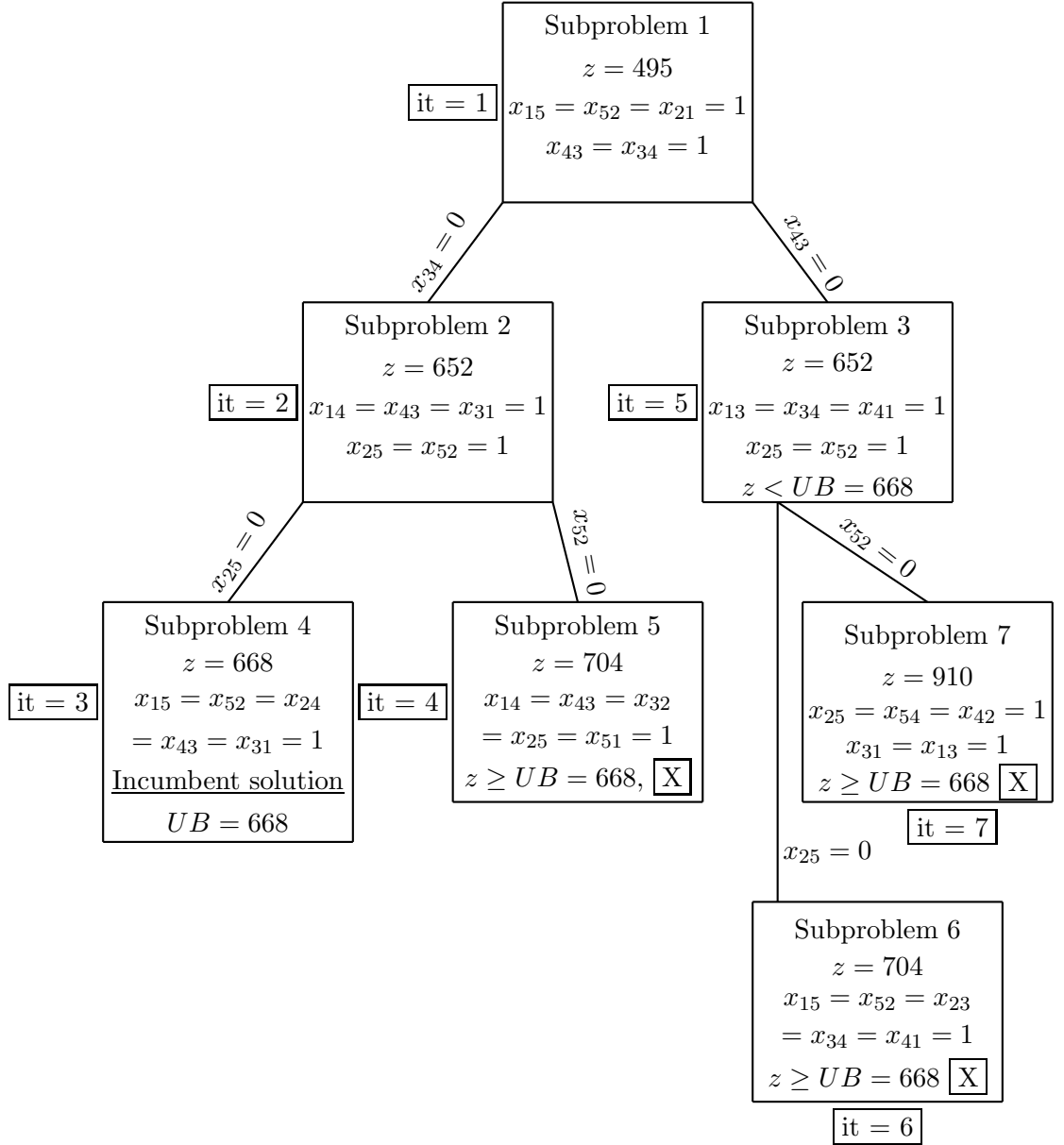
We now solve Subproblem 5. Its cost matrix is

<u>SubP5</u>	City 1	City 2	City 3	City 4	City 5
City 1	M	132	217	164	58
City 2	132	M	290	201	79
City 3	217	290	M	M	303
City 4	164	201	113	M	196
City 5	58	M	303	196	M

Applying the Hungarian method to this cost matrix we obtain an optimal solution $x_{14} = x_{43} = x_{32} = x_{25} = x_{51} = 1$. This

solution is a tour, but $z_{\min} = 704 > UB = 668$. Thus Subproblem 5 is eliminated from consideration.

Other subproblems can be solved similarly. Below is the branching tree diagram of the TSP.



So an optimal solution to the considered TSP is $x_{15} = x_{52} = x_{24} = x_{43} = x_{31} = 1$ with $z_{\max} = 668$, obtained at Subproblem 4.

TSP belongs in the class of combinatorial optimisation problems known as NP-complete. Specifically, if one can find an efficient algorithm (i.e., an algorithm that is guaranteed to find an optimal solution in a polynomial number of steps) for any NP-complete problem, then efficient algorithms exist for all other problems in the NP-complete class. To date, however, no polynomial-time algorithm for any NP-complete problem has been proved.

4 The Cutting Plane Algorithm

Similar to B&B methods, the cutting plane methods for pure or mixed IPs work by solving the LP relaxation of the given IP. The theory of LP dictates that under mild assumptions (if at least an optimal solution exists) one can always find an extreme point or a corner point that is optimal. The obtained optimum to LP relaxation is tested for being an integer solution. If not, it is guaranteed that there exists a linear inequality separating this non-integer optimum from the feasible set of the given IP. To do the separation, we find such an inequality which is called a *cut*. A cut can be added to the LP relaxation to generate an LP relation with a shrunk feasible region. Then, the current non-integer optimal solution is no longer feasible to the cut LP relaxation, which is a new LP relaxation subproblem. This process is repeated until an optimal solution with integer values is found.

Consider an IP problem in standard form

$$\begin{aligned} \max (\text{or } \min) \quad & z = \mathbf{c}^T \mathbf{x} \\ \text{s.t.} \quad & \mathbf{Ax} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \text{ and integer.} \end{aligned}$$

The cutting plane method proceeds by first dropping the requirement that decision variables $\mathbf{x}$ be integers and solving the associated LP-relaxation problem. Geometrically, the optimal solution to the LP relaxation, which is a bfs, is a vertex of the convex polygon consisting of all feasible points for the LP relaxation. If the optimal-solution vertex is not an integer point, then we find a hyperplane such that the vertex is located on one side and all the feasible integer point(s) are on the other side. It is then introduced as an additional linear constraint to exclude the vertex, and a modified LP relaxation subproblem is generated. The generated LP subproblem is then solved and the process is repeated until an integer optimal solution is found.

Step 1. Using the Simplex method to solve an LP relaxation subproblem produces a set of equations of the form

$$x_i + \sum_j \bar{a}_{ij} x_j = \bar{b}_i, \quad (\dagger)$$

where x_i is a basic variable, the x_j 's are the nonbasic variables, $\bar{a}_{ij}$'s are the constraint coefficients in the final optimal tableau, and $\bar{b}_i$ is the rhs in the final optimal tableau.

Step 2. Pick a constraint (i.e. a row) in the final optimal tableau whose rhs is not an integer, i.e. $\bar{b}_i - \lfloor \bar{b}_i \rfloor > 0$.^{*} This constraint will be used to generate a cut. Assume that the i^{th} constraint has been picked. Rewrite this equation ($\dagger$) so that the terms with integer coefficients are on the left side and the terms with fractional coefficients are on the right side:

$$x_i + \sum_j \lfloor \bar{a}_{ij} \rfloor x_j - \lfloor \bar{b}_i \rfloor = (\bar{b}_i - \lfloor \bar{b}_i \rfloor) - \sum_j (\bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor) x_j$$

Since the above equation stems from the equation ($\dagger$), all feasible solutions to the current LP relaxation subproblem satisfy it. Also, since $\bar{b}_i - \lfloor \bar{b}_i \rfloor < 1$, $\bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor \geq 0$, and $x_j \geq 0$, the right side of this equation is strictly less than 1. Hence, for any integer point (i.e. x_i, x_j 's are integers) in the feasible region, the left side of this equation is integer and thus the right side of this equation is less than or equal to 0. So the inequality

$$(\bar{b}_i - \lfloor \bar{b}_i \rfloor) - \sum_j (\bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor) x_j \leq 0 \quad (\ddagger)$$

must hold for any integer point in the feasible region. We add inequality ($\ddagger$) as a new constraint into the current LP relaxation subproblem. It is called the **cut**. Note that

- any feasible integer point for the current LP relaxation subproblem, i.e. any integer solution to the original IP, satisfies the cut, and

^{*}In practice, a constraint in the LP relaxation's optimal tableau whose rhs has the fractional part closest to $\frac{1}{2}$ would be picked.

- the optimal solution to the LP relaxation subproblem, which is not an integer solution with non-integer b_i , **does not** satisfy the cut since all nonbasic variables $x_j = 0$ and hence the inequality (‡) becomes

$$\bar{b}_i - \lfloor \bar{b}_i \rfloor \leq 0, \text{ which is impossible since } \bar{b}_i - \lfloor \bar{b}_i \rfloor > 0.$$

So this cut excludes the optimal solution, which is a bfs as well as an extreme point, of the LP relaxation, and thus is the one with the desired properties. Introducing a new slack variable x_k for the cut (‡), a new equality constraint is added to the LP-relaxation subproblem, namely

$$-\sum (\bar{a}_{ij} - \lfloor a_{ij} \rfloor)x_j + x_k = -(\bar{b}_i - \lfloor \bar{b}_i \rfloor), \text{ and } x_k \geq 0.$$

Step 3. Since the added constraint has a negative rhs, we use the dual Simplex method to find an optimal solution to the modified LP relaxation subproblem (with the cut as an additional constraint). If all variables assume integer values in the optimal solution yielded, we have found an optimal solution to the IP. Otherwise, pick the constraint whose rhs has the fractional part closest to $\frac{1}{2}$ and use it to generate another cut, which is to be added to the current optimal tableau. We continue this process until we obtain a solution in which all variables are integers, i.e. an optimal solution to the given IP.

Example 6. Solve the following IP problem using the cutting-plane algorithm.

$$\begin{aligned}
\max \quad & z = 8x_1 + 5x_2 \\
\text{s.t.} \quad & x_1 + x_2 \leq 6 \\
& 9x_1 + 5x_2 \leq 45 \\
& x_1, x_2 \geq 0 \text{ and integer.}
\end{aligned}$$

Solution. Slack variables s_1 and s_2 , are added to generate the standard form

$$\begin{aligned}
\max \quad & z = 8x_1 + 5x_2 \\
\text{s.t.} \quad & x_1 + x_2 + s_1 = 6 \\
& 9x_1 + 5x_2 + s_2 = 45 \\
& x_1, x_2, s_1, s_2 \geq 0 \text{ and integer.}
\end{aligned}$$

Applying the Simplex method gives the optimal Simplex tableau below.

basis	x_1	x_2	s_1	s_2	rhs
z	0	0	1.25	0.75	41.25
x_2	0	1	2.25	-0.25	2.25
x_1	1	0	-1.25	0.25	3.75

To apply the cutting plane method, we begin by choosing any constraint in the LP-relaxation optimal tableau in which the basic variable is fractional. As both basic variables are equally close to an integer value plus $\frac{1}{2}$, we can arbitrarily choose either to perform the cutting. Here we select the second constraint and rewrite it by taking all decimal parts to rhs

$$x_1 - 1.25s_1 + 0.25s_2 = 3.75$$

$$\Rightarrow x_1 - 2s_1 + 0s_2 - 3 = 0.75 - 0.75s_1 - 0.25s_2.$$

So, the *cut* to be added to the LP-relaxation subproblem is

$$0.75 - 0.75s_1 - 0.25s_2 \leq 0$$

Notice that the optimal solution to the current LP relaxation with $s_1 = s_2 = 0$ does not satisfy this constraint. Adding a new slack variable s_3 to the cut, we obtain a new equality constraint:

$$-0.75s_1 - 0.25s_2 + s_3 = -0.75$$

Now add this row to the final tableau of the LP relaxation with an extra basic variable s_3 and one pivot column for it. Since the rhs of the new constraint is negative, the infeasibility is incurred. To continue solving the problem, the dual Simplex method shall be applied.

basis	x_1	x_2	s_1	s_2	s_3	rhs
z	0	0	1.25	0.75	0	41.25
x_2	0	1	2.25	-0.25	0	2.25
x_1	1	0	-1.25	0.25	0	3.75
s_3	0	0	-0.75	-0.25	1	-0.75
z	0	0	0	0.33	1.67	40
x_2	0	1	0	-1	3	0
x_1	1	0	0	0.67	-1.67	5
s_1	0	0	1	0.33	-1.33	1

One-step dual Simplex procedure gives an optimal solution $(x_1, x_2) = (5, 0)^T$, which is an integer solution. So it is an optimal solution to the original IP, with $z_{\max} = 40$.

Example 7. Solve the following IP problem using the cutting-plane algorithm.

$$\begin{aligned}
\max \quad & z = 7x_1 + 9x_2 \\
\text{s.t.} \quad & -x_1 + 3x_2 \leq 6 \\
& 7x_1 + x_2 \leq 35 \\
& x_1, x_2 \geq 0 \text{ and integer.}
\end{aligned}$$

Solution. Slack variables s_1 and s_2 , are added to generate the standard form. The Simplex method gives an optimal tableau below.

basis	x_1	x_2	s_1	s_2	rhs
z	0	0	$\frac{28}{11}$	$\frac{15}{11}$	63
x_2	0	1	$\frac{7}{22}$	$\frac{1}{22}$	$\frac{7}{2}$
x_1	1	0	$-\frac{1}{22}$	$\frac{3}{22}$	$\frac{9}{2}$

As both basic variables are equally close to an integer value plus $\frac{1}{2}$, we can arbitrarily choose either to perform the cutting. Here we select the first constraint

$$x_2 + \frac{7}{22}s_1 + \frac{1}{22}s_2 = \frac{7}{2}$$

$$\Rightarrow x_2 + 0s_1 + 0s_2 - 3 = \frac{1}{2} - \frac{7}{22}s_1 - \frac{1}{22}s_2.$$

So, the *cut* to be added to the original IP is

$$\frac{1}{2} - \frac{7}{22}s_1 - \frac{1}{22}s_2 \leq 0$$

Adding a new slack variable s_3 to the cut, we obtain a new equality constraint:

$$-\frac{7}{22}s_1 - \frac{1}{22}s_2 + s_3 = -\frac{1}{2}$$

This row now is added to the final tableau of the LP relaxation with an extra basic variable s_3 and one pivot column for it. Since the rhs of the new constraint is negative, the infeasibility is introduced. To continue solving the problem, the dual Simplex method shall be applied.

basis	x_1	x_2	s_1	s_2	s_3	rhs
z	0	0	$\frac{28}{11}$	$\frac{15}{11}$	0	63
x_2	0	1	$\frac{7}{22}$	$\frac{1}{22}$	0	$\frac{7}{2}$
x_1	1	0	$-\frac{1}{22}$	$\frac{3}{22}$	0	$\frac{9}{2}$
s_3	1	0	$-\frac{7}{22}$	$-\frac{1}{22}$	1	$-\frac{1}{2}$
z	0	0	0	1	8	59
x_2	0	1	0	0	1	3
x_1	1	0	0	$\frac{1}{7}$	$-\frac{1}{7}$	$\frac{32}{7}$
s_1	0	0	1	$\frac{1}{7}$	$-\frac{22}{7}$	$\frac{11}{7}$

As the value of x_1 is not an integer, we need to add another cut

$$-\frac{1}{7}s_2 - \frac{6}{7}s_3 + s_4 = -\frac{4}{7}$$

to the above LP relaxation. This gives

basis	x_1	x_2	s_1	s_2	s_3	s_4	rhs
z	0	0	0	1	8	0	59
x_2	0	1	0	0	1	0	3
x_1	1	0	0	$\frac{1}{7}$	$-\frac{1}{7}$	0	$\frac{32}{7}$
s_1	0	0	1	$\frac{1}{7}$	$-\frac{22}{7}$	0	$\frac{11}{7}$
s_4	0	0	0	$-\frac{1}{7}$	$-\frac{6}{7}$	1	$-\frac{4}{7}$
z	0	0	0	0	2	7	55
x_2	0	1	0	0	1	0	3
x_1	1	0	0	0	-1	1	4
s_1	0	0	1	0	-4	1	1
s_2	0	0	0	1	6	-7	4

So, an optimal solution $(x_1, x_2) = (4, 3)^T$ with $z_{\max} = 55$ is obtained for the original IP.

Cutting planes were proposed by Ralph Edward Gomory, an American applied mathematician, in the 1950s as a method for solving pure and mixed IP problems. However, most experts, including Gomory himself, considered them to be impractical due to numerical instability, as well as ineffectiveness because many rounds of cuts are needed to make progress towards the solution. Things turned around when in the mid-1990s Cornuejols and co-workers showed them to be very effective in combination with branch-and-cut and other ways to overcome numerical instabilities.

Further reading: Section 9.6 and 9.8 in the reference book “Operations Research: Applications and Algorithms” (Winston, 2004)