Lecture Notes – Part 3

The Simplex Method and its Algebraic Background

1 Introduction to the Simplex Method

1.1 Background

The Simplex method was devised in 1947 by George Dantzig. It was a remarkably successful technique for solving LPs, and no other solution approach was really considered until the 1980s when interior point methods were devised. This sparked new interest in improving implementations of the Simplex method and today, on most problems, its modified variant is approximately as quick as the best interior point methods.

2.2 Simplex Algorithm Procedure

The Simplex method uses the optimality criterion (cf. Step 2 below) to perform an efficient search of the extreme points (i.e. bfs) of the feasible region. The method usually starts from the bfs where all original decision variables are zeros. Then it "greedily" (in the sense that the objective function value is getting improved) moves from one extreme point (i.e. bfs) of the feasible region to an *adjacent* bfs[5] by changing one basic variable at a time. Intuitively, two bfs are adjacent if they both lie on the same edge of the boundary of the feasible region. In the searching/moving procedure, the feasibility criterion (cf. Step 3 below), which is a ratio test, ensures that the basic solution in each iteration remains feasible (i.e. satisfies all constraints). The method ceases when no further improvement in the value of the objective function

[5]For any LP with m constraints, two bfs are said to be "adjacent" if their bases have $m-1$ basic variables in common.

can be obtained, and an optimal solution, if it exists, is exactly the most recent bfs, i.e. the current bfs. The method is outlined below:

Step 1: (Preprocessing – Initial bfs) Convert the LP to standard form. Check whether the system of linear equations is in *canonical form* where each equation has a variable with a coefficient of 1 in that equation and zero coefficients in all other equations. If this is the case and the rhs of each equation in the canonical form is nonnegative, an initial bfs can be obtained by inspection, i.e. a basis which is feasible can be constructed by those variables with a coefficient of 1 in one equation and a zero coefficient in any other equation. Then generate an initial Simplex tableau from the objective and equations written in canonical form. Following the notation in the text (Winston, 2004), the objective function row, also called the reduced cost row or row 0, is constructed by moving the rhs variable terms of the objective, i.e. $\mathbf{c}^T\mathbf{x}$ to the lhs (so the constant term, usually 0, will be on the rhs).[6] The reason why we move all the variable terms to the lhs for the objective function row is obvious. To get a Simplex tableau with consistent columns, we shall let all the variables stay on the lhs as the system of constraints $\mathbf{Ax} = \mathbf{b}$. Thus, we shall have row 0 as $z - \mathbf{c}^T\mathbf{x} = 0$ instead of $z = \mathbf{c}^T\mathbf{x}$.

Step 2: (Entering) For a maximisation LP, if there is no negative number in the row 0 (the objective function row), then STOP – the current bfs is optimal. Otherwise, select a nonbasic variable with the most negative number (which is called reduced cost) in row 0 to be the *entering variable* x_t, which will become basic variable after iteration. For minimisation, if there is no positive entry in row 0, then STOP – the current bfs is optimal. Otherwise, select a nonbasic variable with the largest positive reduced cost as the entering variable.

Step 3: (Leaving – Ratio Test) Let $\widehat{\mathbf{b}} = (\widehat{b}_1, \widehat{b}_2, \cdots, \widehat{b}_m)^T$ be the right-most column, which is called "column rhs", and let $\widehat{\mathbf{A}}_t = (\widehat{a}_{1t}, \widehat{a}_{2t}, \cdots, \widehat{a}_{mt})^T$ be the column vector corresponding to the entering variable x_t. Find an index

[6] In the representation of the Simplex tableau in the text, an additional column is added to the tableau. This column has the same entries regardless of the row operations, a 1 followed by 0s. Hence, in most representations this column can be omitted. This will be the case in our lecture notes.

$$s = \arg \min_{1 \le i \le m} \left\{ \frac{\widehat{b}_i}{\widehat{a}_{it}} : \widehat{a}_{it} > 0 \right\}$$

i.e. find the smallest "positive" ratio formed by the divisor of column rhs and the entering-variable column. The basic variable with the smallest positive ratio is the *leaving variable*, i.e. if

$$\min_{1 \le i \le m} \left\{ \frac{\widehat{b}_i}{\widehat{a}_{it}} : \widehat{a}_{it} > 0 \right\} = \frac{\widehat{b}_s}{\widehat{a}_{st}}$$

then x_s is the leaving variable.

If $\widehat{a}_{it} \le 0$ for all $i = 1, 2, \cdots, m$, then STOP – the problem is unbounded.

Step 4: (Pivoting) Update the tableau by pivoting on a_{st}, i.e. perform EROs on the tableau to get a 1 in the pivot position, and 0s above and below it.[7] Recall that the pivot is the intersection of the entering-variable column and the leaving-variable row. By doing this, you are solving the system of linear equations with the updated nonbasic variables being set zeros. This tableau then yields the new bfs. The process then returns to **Step 2** to commence the next iteration, if necessary.

An n-variable, m-constraint LP can have at most $\binom{n}{m} = \frac{n!}{m!(n-m)!}$ basic solutions, and therefore bfs. The Simplex method is a process of searching an optimal bfs by moving from a bfs to an adjacent one. This means (assuming that no bfs is repeated) that the Simplex method will find an optimal bfs after a finite number of iterations. In principle, we could enumerate all bfs to an LP and find the bfs with the best z-value. The problem with this approach is that even small-size LPs have a very large number of bfs. For example, an LP in standard form that has 20 variables and 10 constraints might have (if each basic solution were feasible) up to $\binom{20}{10} = 184,756$ bfs. Fortunately, vast experience with the Simplex algorithm indicates that when this algorithm is applied to an n-variable, m-constraint LP in standard form, an optimal solution is usually found after examining fewer than $3m$ bfs. Thus, for a

[7]The pivot element a_{st} is always a "positive" number since **Step 3** ignores any negative elements in the entering-variable column. Also, all the elements in the rhs remain nonnegative. (At this stage, notice that if you find any negative rhs entry, it's a sure sign that you've made a mistake somewhere during the Simplex procedure! More complicated situations will be addressed in Chapter 7.)

20-variable, 10-constraint LP in standard form, the Simplex procedure will usually find an optimal solution after examining fewer than 30 bfs. Compared with the alternative of examining $184,756$ basic solutions, the Simplex method i s quite efficient.

2 The Simplex Method in Algebraic Form

In the previous chapter, the fundamental law of LP, a concise introduction of basic feasible solution, and a brief demonstration of the Simplex method in tabular form had been given. In this chapter, we introduce the Simplex method in algebraic form, which is essential to the sensitivity analysis and duality theory of LP. The more detailed theoretical background of the Simplex algorithm will be provided.

Recall that the Simplex method is an iterative method for solving an LP in standard form. It moves from a basic feasible solution (bfs) to another with a better objective value until an optimal solution is found, if it exists.[1]

Before introducing the algebraic formulae of the Simplex method, the algebraic Simplex procedure will be illustrated and explained by an example of a minimisation LP. Then the connection of the Simplex algorithm in algebraic form and the Simplex tableau will be made.

2.1 Example of Algebraic Simplex Procedure

Consider the following LP:

$$
\begin{array}{rrcl}
\min z = & -x_1 - 2x_2 & & \\
s.t. & -2x_1 + x_2 & \leq & 2 \\
 & -x_1 + 2x_2 & \leq & 7 \\
 & x_1 & \leq & 3 \\
 & x_1, \quad x_2 & \geq & 0
\end{array}
$$

[1]Other situations such as unboundedness, infeasibility and degeneracy will be introduced in this and the following chapters.

To convert it to the standard form, three slack variables x_3, x_4 and x_5 are introduced as follows.[2]

$$
\begin{array}{rrcrcrcrcrcl}
\min z = & -x_1 & - & 2x_2 & & & & & & & & \\
s.t. & -2x_1 & + & x_2 & + & x_3 & & & & & = & 2 \\
 & -x_1 & + & 2x_2 & & & + & x_4 & & & = & 7 \\
 & x_1 & & & & & & & + & x_5 & = & 3 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5 & \geq & 0
\end{array}
$$

The matrix form of this LP in standard form is

$$
\begin{array}{rrcl}
\min\ z = & \mathbf{c}^T\mathbf{x} & & \\
s.t. & \mathbf{A}\mathbf{x} & = & \mathbf{b} \\
 & \mathbf{x} & \geq & \mathbf{0},
\end{array}
$$

where

$$
\begin{aligned}
&\mathbf{x} = (x_1,\ x_2,\ x_3,\ x_4,\ x_5)^T, \\
&\mathbf{c}^T = (-1, -2,\ 0,\ 0,\ 0), \\
&\mathbf{b} = (2,\ 7,\ 3)^T \\
&\mathbf{A} = \begin{pmatrix} -2 & 1 & 1 & 0 & 0 \\ -1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}.
\end{aligned}
$$

Each of the constraints has a unique slack variable. Obviously, these slack variables can be chosen as the basic variables to construct an initial basis, i.e. $\mathbf{x_B} = (x_3, x_4, x_5)^T$. By setting the other part of decision variables, nonbasic variables, equal to zero, i.e. $\mathbf{x_N} = (x_1, x_2)^T = \mathbf{0}$, and solving the system of linear equations, we obtain the initial bfs

$$(x_1,\ x_2,\ x_3,\ x_4,\ x_5) = (0,\ 0,\ 2,\ 7,\ 3).$$

The corresponding objective function value is $z = 0$.

We now investigate whether there exists an adjacent bfs, which can be obtained by changing **only one** element in the current basis, and give a better objective value.

[2] To facilitate further presentation, we set the slack variables with the notation x_i instead of s_i.

To do this, we first express each basic variable in $\mathbf{x_B}$ in terms of the nonbasic variable(s) x_1 and/or x_2 as follows:

$$\begin{aligned} x_3 &= 2 + 2x_1 - x_2, \\ x_4 &= 7 + x_1 - 2x_2, \\ x_5 &= 3 - x_1, \end{aligned}$$

and rewrite the objective in the equality form with the constant term left on the rhs:

$$z + x_1 + 2x_2 = 0.$$

Recall that the current bfs retains nonbasic variables $x_1 = x_2 = 0$ and the objective function value $z = 0$. Now observe that the coefficients of x_1 and x_2 in the objective equality form are positive. If either x_1 or x_2 is increased from zero, then the z-value will decrease, meaning that a better solution will be yielded. Since the coefficient of x_2, 2, is greater than that of x_1, 1, the z-value will decrease more rapidly by increasing x_2 than x_1. With a greedy attitude to the progress on the solution, we prefer to choose x_2 as the new element (entering variable) in the new basis.[3] Then x_1 is still a nonbasic variable, whose value in the new basic solution is kept as zero.

Now by taking $x_1 = 0$, the objective in the equality form becomes

$$z + 2x_2 = 0,$$

and the constraints become

$$\begin{aligned} x_3 &= 2 - x_2, \\ x_4 &= 7 - 2x_2, \\ x_5 &= 3. \end{aligned}$$

The more x_2 is increased, the more improved the z-value gets. However, to maintain the nonnegativity of the basic variables the value of x_2 must satisfy

$$\begin{aligned} x_3 = 2 - x_2 \geq 0 \quad &\Rightarrow \quad x_2 \leq \tfrac{2}{1} \\ x_4 = 7 - 2x_2 \geq 0 \quad &\Rightarrow \quad x_2 \leq \tfrac{7}{2} \end{aligned}$$

[3]Choosing the nonbasic variable with the most positive coefficient (for the minimisation LP) in row 0 as the entering variable usually but not always leads us quickly to the optimal bfs. Actually, even if we choose x_1, the Simplex algorithm will eventually find an optimal solution, though, probably with more iterations.

Hence we have

$$x_2 \le \min\left\{\frac{2}{1}, \frac{7}{2}\right\} = 2.$$

So x_2 can only be increased from 0 to 2. By doing so, we have

$$\begin{aligned} x_3 &= 2 - 2 = 0 \text{ (the leaving variable)}, \\ x_4 &= 7 - 2\times 2 = 3. \end{aligned}$$

Hence we obtain a new bfs

$$(x_1,\ x_2,\ x_3,\ x_4,\ x_5) = (0,\ 2,\ 0,\ 3,\ 3).$$

The new basis is (x_2, x_4, x_5). The corresponding objective value is $z = -2 \times 2 = -4$. This completes the first iteration of the Simplex method.

Now the second iteration can commence with checking whether the current basis (x_2, x_4, x_5) is optimal.
Again, we express the basic variables in terms of nonbasic variables (x_1, x_3):

$$\begin{aligned} x_2 &= 2 + 2x_1 - x_3, \\ x_4 &= 7 + x_1 - 2x_2 \\ &= 7 + x_1 - 2(2 + 2x_1 - x_3) \\ &= 3 - 3x_1 + 2x_3, \\ x_5 &= 3 - x_1; \end{aligned}$$

Or equivalently,

$$\begin{array}{rrrrrl} -2x_1 & +x_2 & +x_3 & & & = 2, \\ 3x_1 & & -2x_3 & +x_4 & & = 3, \\ x_1 & & & & +x_5 & = 3. \end{array}$$

And we set the objective in the equality form also in terms of nonbasic variables (x_1, x_3) with constant term left on rhs.

$$\begin{array}{rlcr} & z + x_1 + 2x_2 & = & 0 \\ \Rightarrow & z + x_1 + 2(2 + 2x_1 - x_3) & = & 0 \\ \Rightarrow & z + 4 + 5x_1 - 2x_3 & = & 0 \\ \Rightarrow & z + 5x_1 - 2x_3 & = & -4 \end{array}$$

Now observe that in the above objective equality form only the coefficient of x_1 is positive. To get a better objective function value, we can only increase x_1. So x_3 is still a nonbasic variable, whose value in the new basic solution is kept as 0. By taking $x_3 = 0$, the objective in the equality form becomes

$$z + 5x_1 \;=\; -4,$$

and the constraints become

$$\begin{aligned} x_2 &= 2 + 2x_1, \\ x_4 &= 3 - 3x_1, \\ x_5 &= 3 - x_1. \end{aligned}$$

To maintain the nonnegativity of the basic variables, the value of x_1 must satisfy

$$\begin{aligned} x_2 = 2 + 2x_1 \geq 0 \quad &\Rightarrow \quad x_1 \geq -\tfrac{2}{2} \text{ (redundant)} \\ x_4 = 3 - 3x_1 \geq 0 \quad &\Rightarrow \quad x_1 \leq \tfrac{3}{3} \\ x_5 = 3 - x_1 \geq 0 \quad &\Rightarrow \quad x_1 \leq \tfrac{3}{1} \end{aligned}$$

Hence

$$x_1 \leq \min\left\{\frac{3}{3}, \frac{3}{1}\right\} = 1.$$

Thus x_1 can only be increased from 0 to 1. By doing so, we have

$$\begin{aligned} x_2 &= 2 + \;\; 2 \quad = 4, \\ x_4 &= 3 - 3 \times 1 = 0 \text{ (the leaving variable)}, \\ x_5 &= 3 - \;\; 1 \quad = 2. \end{aligned}$$

And it yields a new bfs

$$(x_1,\ x_2,\ x_3,\ x_4,\ x_5) = (1,\ 4,\ 0,\ 0,\ 2).$$

Now the new basis is $\mathbf{x_B} = (x_1, x_2, x_5)^T$, and the corresponding objective value is $z = -4 - 5 \times 1 = -9$. This completes the second iteration. The next iteration can be continued using the current basis $\mathbf{x_B} = (x_1, x_2, x_5)$. We stop the demonstration of this example here.

Note that at each iteration we identify only one nonbasic variable that can improve the objective, if it exists. Then the entering variable is increased until some basic variable decreases to zero and leaves the basis. This gives a new bfs, and the process repeats until an optimal bfs is yielded.

2.2 General Formulae of the Simplex Method

The Simplex method starts with a bfs (extreme point), and then searches for a better one. If no bfs with a better objective value can be found, then we are at an optimal solution. Otherwise, it steps to a new bfs and repeats the process. The Simplex search is moving from one extreme point to an adjacent extreme point[4] along an "edge" of the feasible region. In particular, only one of the nonbasic variables is allowed to move away from 0 and enter the basis; at the same time one of the basic variables becomes 0 and leaves the basis. That is, we change **only one** element of the basis at each iteration.

In this section, we present the general algebraic formulae of the Simplex method, assuming that an initial bfs[5] exists.
Consider a **minimisation** LP in standard form

$$\begin{aligned} \min z = \quad & \mathbf{c}^T\mathbf{x} & & \\ s.t. \quad & \mathbf{A}\mathbf{x} & = & \mathbf{b} \\ & \mathbf{x} & \geq & \mathbf{0}, \end{aligned}$$

where

$$\begin{aligned} \mathbf{x} &= (x_1, \cdots, x_n)^T; \\ \mathbf{c} &= (c_1, \cdots, c_n)^T; \\ \mathbf{b} &= (b_1, \cdots, b_m)^T, \quad b_i \geq 0, \forall i = 1, \dots, m; \end{aligned}$$

$$\mathbf{A} = [\mathbf{A}_1, \cdots, \mathbf{A}_n] = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \dots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$$

with $n > m$ and $rank(\mathbf{A}) = m$.

From $\mathbf{x}$, we select a set of m variables whose corresponding column vectors in matrix $\mathbf{A}$ are linearly independent to construct a basis $\mathbf{x_B}$. Denote by $\mathbf{B}$ the $m \times m$ matrix which is defined by those columns, and call it *basic matrix*. Recall that the variables in the basis $\mathbf{x_B}$ are called the basic variables.
The remaining $n - m$ variables in $\mathbf{x}$ are called the nonbasic variables, and the vector constructed with them is denoted by $\mathbf{x_N}$ and called the nonbasis. The $m \times (n-m)$ matrix built by their respective columns in $\mathbf{A}$ is denoted by

[4] Recall that for any m-constraint LP two bfs are adjacent if their bases share $m - 1$ common basic variables.

[5] We will show how to find an initial bfs shortly.

$\mathbf{N}$ and called *nonbasic matrix*. Without loss of generality[6], we assume that the basic variables in $\mathbf{x_B}$ are the first m entries in $\mathbf{x}$, i.e.

$$\mathbf{x}^T = (\mathbf{x_B}^T | \mathbf{x_N}^T),$$

and hence

$$\mathbf{A} = [\mathbf{B}|\mathbf{N}].$$

The corresponding coefficients of $\mathbf{x_B}$ and $\mathbf{x_N}$ in the objective function are denoted by $\mathbf{c_B}$ and $\mathbf{c_N}$, respectively, i.e.

$$\mathbf{c}^T = (\mathbf{c_B}^T | \mathbf{c_N}^T).$$

Taking the aforementioned example with $m = 3$ and $n = 5$ for instance, we have

$$\begin{aligned} \mathbf{c}^T &= (-1,\ -2,\ 0,\ 0,\ 0), \\ \mathbf{A} &= \begin{pmatrix} -2 & 1 & 1 & 0 & 0 \\ -1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}, \\ \mathbf{b}^T &= (2,\ 7,\ 3). \end{aligned}$$

If we choose the basis $\mathbf{x_B} = (x_3,\ x_4,\ x_5)^T$, then we have

$$\mathbf{x_N} = (x_1,\ x_2)^T,$$

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathbf{N} = \begin{pmatrix} -2 & 1 \\ -1 & 2 \\ 1 & 0 \end{pmatrix},$$

$$\mathbf{c_B}^T = (0,\ 0,\ 0), \qquad \mathbf{c_N}^T = (-1,\ -2).$$

Using the notation, the considered LP can be rewritten in the form

$$\begin{aligned} \min z = \quad & \mathbf{c_B}^T\mathbf{x_B} + \mathbf{c_N}^T\mathbf{x_N} \\ s.t. \quad & \mathbf{B}\mathbf{x_B} + \mathbf{N}\mathbf{x_N} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}. \end{aligned}$$

Or equivalently, we find the minimum value of z satisfying the condition

$$z - \mathbf{c_B}^T\mathbf{x_B} - \mathbf{c_N}^T\mathbf{x_N} = 0, \tag{1}$$

[6]The order of components in $\mathbf{x}$ can be rearranged, if necessary.

subject to

$$\mathbf{B}\mathbf{x_B} + \mathbf{N}\mathbf{x_N} = \mathbf{b}, \tag{2}$$

$$\mathbf{x} \geq \mathbf{0}. \tag{3}$$

Recall that if the basis $\mathbf{x_B}$ obtained by setting $\mathbf{x_N} = \mathbf{0}$ in (2) satisfies (3), i.e. $\mathbf{x_B} = \mathbf{B}^{-1}\mathbf{b} \geq \mathbf{0}$, then the basic solution $(\mathbf{x_B}^T|\mathbf{0}^T)$ is feasible and called basic feasible solution (bfs).

To facilitate the further observation of the Simplex process of moving from one bfs to an adjacent bfs, i.e. swapping one nonbasic variable in the nonbasis $\mathbf{x_N}$ for one basic variable in the basis $\mathbf{x_B}$ in each iteration, we reformulate the basic variables and the objective in terms of the nonbasic variables.

From (2) we can express $\mathbf{x_B}$ in terms of $\mathbf{x_N}$ as follows:

$$\begin{aligned} \mathbf{B}\mathbf{x_B} &= \mathbf{b} - \mathbf{N}\mathbf{x_N}, \\ \mathbf{x_B} &= \mathbf{B}^{-1}(\mathbf{b} - \mathbf{N}\mathbf{x_N}) \\ &= \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x_N}. \end{aligned} \tag{4}$$

To express the objective function in terms of $\mathbf{x_N}$, we substitute (4) into (1) and have

$$\begin{aligned} z - \left(\mathbf{c_B}^T(\mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x_N}) + \mathbf{c_N}^T\mathbf{x_N}\right) &= 0, \\ z - \mathbf{c_B}^T\mathbf{B}^{-1}\mathbf{b} + (\mathbf{c_B}^T\mathbf{B}^{-1}\mathbf{N} - \mathbf{c_N}^T)\mathbf{x_N} &= 0. \end{aligned}$$

So, the value of z must satisfy the equality

$$z + (\mathbf{c_B}^T\mathbf{B}^{-1}\mathbf{N} - \mathbf{c_N}^T)\mathbf{x_N} = \mathbf{c_B}^T\mathbf{B}^{-1}\mathbf{b}, \tag{5}$$

and the rhs of (5) is a constant term.

Denote

$$\widehat{\mathbf{c}}_\mathbf{N}^T = \mathbf{c_B}^T\mathbf{B}^{-1}\mathbf{N} - \mathbf{c_N}^T.$$

Each component $\widehat{c}_j$ of $\widehat{\mathbf{c}}_\mathbf{N}$ is called the *reduced cost*[7] of the nonbasic variable x_j in $\mathbf{x_N}$.

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[7]The reduced cost of a decision variable is its coefficient in row 0 of the Simplex tableau. Note that the reduced costs for all basic variables are always 0s. The reduced cost of a nonbasic variable is the amount by which the value of z will "decrease" if we increase the value of the nonbasic variable by 1 (while all the other nonbasic variables remain equal to 0). In other words, it indicates how much the objective function coefficient on the corresponding variable must be improved before the value of the variable will be positive in the optimal solution.

Then the equality (5) can be rewritten in the form

$$z + \widehat{\mathbf{c}}_{\mathbf{N}}^T \mathbf{x}_{\mathbf{N}} \;=\; \mathbf{c}_{\mathbf{B}}^T \mathbf{B}^{-1}\mathbf{b}. \tag{6}$$

To analyse the possibility of improving the z-value by entering one nonbasic variable in $\mathbf{x}_{\mathbf{N}}$ into the basis, i.e. increasing one nonbasic variable from zero and hence making $\mathbf{x}_{\mathbf{N}} \geq \mathbf{0}$, we consider the following two cases for (6).

Case 1: $\widehat{\mathbf{c}}_{\mathbf{N}} \leq \mathbf{0}$.

From (6) we have

$$z = \mathbf{c}_{\mathbf{B}}^T \mathbf{B}^{-1}\mathbf{b} - \widehat{\mathbf{c}}_{\mathbf{N}}^T \mathbf{x}_{\mathbf{N}},$$

which means that the z-value could increase and will never get improved if we increase any nonbasic variable instead of keeping $\mathbf{x}_{\mathbf{N}} = \mathbf{0}$. If $\mathbf{B}^{-1}\mathbf{b} \geq \mathbf{0}$, the basic solution $(\mathbf{x}_{\mathbf{B}}^T|\mathbf{0}^T)$ is a bfs and hence an optimal solution with the optimal objective value $z = \mathbf{c}_{\mathbf{B}}^T \mathbf{B}^{-1}\mathbf{b}$.

Case 2: There exists at least one positive reduced cost in $\widehat{\mathbf{c}}_{\mathbf{N}}$.

In this case, the objective value can be improved by increasing one of the nonbasic variables with positive reduced costs, say x_t. Hence we can generate a new basis by entering x_t into the current basis. And then one original basic variable needs to leave from the current basis.
Once x_t with $\widehat{c}_t > 0$ has been selected, we must determine how much it can be increased before any nonnegativity constraint is violated.
We have the current basic variables defined by (4)

$$\mathbf{x}_{\mathbf{B}} \;=\; \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_{\mathbf{N}}.$$

Setting all components of $\mathbf{x}_{\mathbf{N}}$ other than x_t equal to zeros gives

$$\mathbf{x}_{\mathbf{B}} \;=\; \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{A}_t x_t \;=\; \widehat{\mathbf{b}} - \widehat{\mathbf{A}}_t x_t, \tag{7}$$

where $\widehat{\mathbf{b}} = \mathbf{B}^{-1}\mathbf{b}$ and $\widehat{\mathbf{A}}_t = \mathbf{B}^{-1}\mathbf{A}_t$.
Denoting $\widehat{\mathbf{A}}_t = (\widehat{a}_{1t},\ \widehat{a}_{2t}, \cdots, \widehat{a}_{mt})^T$, we examine (7) component-wise: for each $i = 1, 2, \ldots, m$,

$$(\mathbf{x}_{\mathbf{B}})_i = \widehat{b}_i - \widehat{a}_{it} x_t.$$

Then we have the following two subcases.

(i) $\widehat{\mathbf{A}}_t \leq \mathbf{0}$.

None of the basic variables will decrease in value as x_t is increased from zero. So x_t can be increased unlimitedly without causing the violation of any nonnegativity constraint. In this case, the objective value will decrease and thus get improved unboundedly as $x_t \to \infty$, indicating that the LP is *unbounded*.

(ii) There exists at least one positive component in $\widehat{\mathbf{A}}_t$.

Assume we have $\widehat{a}_{it} > 0$ for some i. Then $(\mathbf{x_B})_i$ will decrease as x_t increases. More precisely, $(\mathbf{x_B})_i$ will decrease to zero when x_t is increased from 0 to $\dfrac{\widehat{b}_i}{\widehat{a}_{it}}$.

Now assume that we have more than one positive components in $\widehat{\mathbf{A}}_t$. Then the value to which x_t can be increased is decided by

$$\min_{1 \leq i \leq m} \left\{ \frac{\widehat{b}_i}{\widehat{a}_{it}} : \widehat{a}_{it} > 0 \right\}.$$

This is exactly the *ratio test*. If the index

$$s = \arg \min_{1 \leq i \leq m} \left\{ \frac{\widehat{b}_i}{\widehat{a}_{it}} : \widehat{a}_{it} > 0 \right\}$$

is obtained, then x_t will take over the position of $(\mathbf{x_B})_s$, which will leave for $\mathbf{x_N}$, to make a new basis and hence a new bfs which gives a better objective value.[8]

2.3 The Simplex Algorithm in Algebraic Form

In this section, we outline the Simplex algorithm in algebraic form.
Consider the LP in standard form

$$\begin{array}{lrcl} \min \text{ (or max) } z = & \mathbf{c}^T\mathbf{x} & & \\ s.t. & \mathbf{A}\mathbf{x} & = & \mathbf{b} \\ & \mathbf{x} & \geq & \mathbf{0}, \end{array}$$

where $\mathbf{b} \geq \mathbf{0}$.

[8] The current nonbasic variable x_t is called the entering variable, and the current basic variable $(\mathbf{x_B})_s$ is called the leaving variable.

The Simplex method starts with a feasible basis $\mathbf{x_B}$ and the corresponding basic matrix $\mathbf{B}$ (i.e. the condition $\mathbf{B}^{-1}\mathbf{b} \geq \mathbf{0}$ holds). Using the notation defined in the last section, the steps of the Simplex algorithm are given below.

Step 1. Compute the vector

$$\widehat{\mathbf{c}}_{\mathbf{N}}^T = \mathbf{c}_{\mathbf{B}}^T\mathbf{B}^{-1}\mathbf{N} - \mathbf{c}_{\mathbf{N}}^T$$

- If $\widehat{\mathbf{c}}_{\mathbf{N}} \leq \mathbf{0}$ (or $\widehat{\mathbf{c}}_{\mathbf{N}} \geq \mathbf{0}$), the current bfs is optimal; STOP.
- Otherwise, select a variable x_t satisfying $\widehat{c}_t > 0$ with the most positive $\widehat{c}_t$ (or $\widehat{c}_t < 0$ with the most negative $\widehat{c}_t$) as the entering variable.

Step 2. Let $\mathbf{A}_t$ be the column of $\mathbf{A}$ corresponding to the variable x_t. Compute $\widehat{\mathbf{A}}_t = \mathbf{B}^{-1}\mathbf{A}_t$. Let $\widehat{\mathbf{A}}_t = (\widehat{a}_{1t},\ \widehat{a}_{2t}, \cdots, \widehat{a}_{mt})^T$.

- If $\widehat{\mathbf{A}}_t \leq \mathbf{0}$, then the LP problem is unbounded.
- Otherwise, find the index

$$s = \arg \min_{1\leq i\leq m} \left\{ \frac{\widehat{b}_i}{\widehat{a}_{it}} : \widehat{a}_{it} > 0 \right\}.$$

  Then select $(\mathbf{x_B})_s$ as the component which is leaving out of the current basis $\mathbf{x_B}$.

Step 3. Replace $(\mathbf{x_B})_s$ with x_t to obtain a new basis $\mathbf{x_B}$. Update the basic matrix $\mathbf{B}$. Then GO TO **Step 1**.

2.4 The Algebraic Simplex Tableau

This section introduces the algebraic Simplex tableau, where the algebraic formula in each entry is derived from the Simplex general formula. The Simplex procedure can be done by iterating the calculation of the algebraic Simplex tableau with a given basis. Let $\mathbf{x_B}$ and $\mathbf{B}$ be respectively the chosen basis and the corresponding basic matrix to initiate some Simplex iteration. In matrix-vector notation, the current full Simplex tableau is of the form:

basis	$\mathbf{x}$	rhs
z	$\mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{A} - \mathbf{c}^T$ $= \widehat{\mathbf{c}}^T$	$\mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{b}$
$\mathbf{x}_\mathbf{B}$	$\mathbf{B}^{-1}\mathbf{A}$ $= \widehat{\mathbf{A}}$	$\mathbf{B}^{-1}\mathbf{b}$ $= \widehat{\mathbf{b}}$

Notice that we have chosen the basic solution $(\mathbf{x}_\mathbf{B}^T|\mathbf{0}^T)$. Hence, from (4) we have $\mathbf{x}_\mathbf{B} = \mathbf{B}^{-1}\mathbf{b}$, which gives the bottom right entry $\widehat{\mathbf{b}}$. Since the system of equation is $\mathbf{A}\mathbf{x} = \mathbf{b}$, we have $\mathbf{B}^{-1}\mathbf{A}\mathbf{x} = \mathbf{B}^{-1}\mathbf{b}$, which accounts for the bottom left entry $\widehat{\mathbf{A}} = \mathbf{B}^{-1}\mathbf{A}$. Since from (1) the objective function is $z = \mathbf{c}_\mathbf{B}^T\mathbf{x}_\mathbf{B} + \mathbf{c}_\mathbf{N}^T\mathbf{x}_\mathbf{N}$, the basic solution $\mathbf{x}_\mathbf{B} = \mathbf{B}^{-1}\mathbf{b}$, $\mathbf{x}_\mathbf{N} = \mathbf{0}$ makes $z = \mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{b}$, which gives the top right entry. To get the top left entry, we return back to the original objective function $z - \mathbf{c}^T\mathbf{x} = 0$ and thus have $z - \mathbf{c}^T\mathbf{x} + \mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{b} = \mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{b}$. Then it is obvious that $z - \mathbf{c}^T\mathbf{x} + \mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{A}\mathbf{x} = z + (\mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{A} - \mathbf{c}^T)\mathbf{x} = \mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{b}$, which gives $\widehat{\mathbf{c}}^T = \mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{A} - \mathbf{c}^T$.

Decomposing the decision variables $\mathbf{x}$ accordingly into the basis $\mathbf{x}_\mathbf{B}$ and non-basis $\mathbf{x}_\mathbf{N}$, we have the equivalent tableau:

basis	$\mathbf{x}_\mathbf{N}$	$\mathbf{x}_\mathbf{B}$	rhs
z	$\mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{N} - \mathbf{c}_\mathbf{N}^T$	$\mathbf{0}^T$	$\mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{b}$
$\mathbf{x}_\mathbf{B}$	$\mathbf{B}^{-1}\mathbf{N}$	$\mathbf{I}$	$\mathbf{B}^{-1}\mathbf{b}$

While it is not realistic to divide the columns into two groups with $\mathbf{x}_\mathbf{B}$ and $\mathbf{x}_\mathbf{N}$ like the above tableau (i.e. continually reordering the columns based on the current basis $\mathbf{x}_\mathbf{B}$), it does illustrate the property that the submatrix in the Simplex tableau corresponding to $\mathbf{x}_\mathbf{B}$ is the identity matrix, and the reduced costs of all the components of $\mathbf{x}_\mathbf{B}$ are zeros.
The ratio test and EROs at each Simplex iteration will be done in each Simplex tableau. Actually, the EROs performed after deciding the entering variable paired with the leaving variable is equivalent to the calculation of the entries in the tableau with the above algebraic formulae. In other words, the above algebraic Simplex tableau can be utilised to generate a Simplex tableau for any given basis $\mathbf{x}_\mathbf{B}$.

Now we look back on the example in Section 2.1 and utilise the Simplex tableau technique to solve it. Its initial Simplex tableau is shown as below.

basis	x_1	x_2	x_3	x_4	x_5	rhs
z	1	2	0	0	0	0
x_3	-2	1	1	0	0	2
x_4	-1	2	0	1	0	7
x_5	1	0	0	0	1	3

Since the most positive reduced cost is $\widehat{c}_2 = 2$, x_2 is the entering variable. The ratio test $\min\{\frac{2}{1}, \frac{7}{2}\}$ gives that x_3 is the leaving variable. By pivoting[9], a new basis and thus bfs can be produced in the second Simplex tableau added below the first one.

basis	x_1	x_2	x_3	x_4	x_5	rhs	
z	1	2	0	0	0	0	
x_3	-2	$\boxed{1}$	1	0	0	2	$\frac{2}{1} = 2$
x_4	-1	2	0	1	0	7	$\frac{7}{2} = 3.5$
x_5	1	0	0	0	1	3	
z	5	0	-2	0	0	-4	$R_0' \leftarrow R_0 - 2R_1$
x_2	-2	1	1	0	0	2	
x_4	3	0	-2	1	0	3	$R_2' \leftarrow R_2 - 2R_1$
x_5	1	0	0	0	1	3	

In the current Simplex tableau, the most positive reduced cost is $\widehat{c}_1 = 5$. So, x_1 is the entering variable. The ratio test $\min\{\frac{3}{3}, \frac{3}{1}\}$ gives that x_4 is the leaving variable. Then again we have a new basis and bfs in the third Simplex tableau. The Simplex procedure is repeated as shown in the tableaux below.

[9]You can perform the pivoting with EROs or do the calculation of entries by Simplex algebraic formulae.

basis	x_1	x_2	x_3	x_4	x_5	rhs	
z	1	2	0	0	0	0	
x_3	-2	$\boxed{1}$	1	0	0	2	
x_4	-1	2	0	1	0	7	
x_5	1	0	0	0	1	3	
z	5	0	-2	0	0	-4	$R_0' \leftarrow R_0 - 2R_1$
x_2	-2	1	1	0	0	2	
x_4	$\boxed{3}$	0	-2	1	0	3	$R_2' \leftarrow R_2 - 2R_1$
x_5	1	0	0	0	1	3	
z	0	0	$\frac{4}{3}$	-$\frac{5}{3}$	0	-9	$R_0'' \leftarrow R_0' - 5R_2''$
x_2	0	1	-$\frac{1}{3}$	$\frac{2}{3}$	0	4	$R_1'' \leftarrow R_1' + 2R_2''$
x_1	1	0	-$\frac{2}{3}$	$\frac{1}{3}$	0	1	$R_2'' \leftarrow \frac{1}{3}R_2'$ (go first)
x_5	0	0	$\boxed{\frac{2}{3}}$	-$\frac{1}{3}$	1	2	$R_3'' \leftarrow R_3' - R_2''$
z	0	0	0	-1	-2	-13	$R_0''' \leftarrow R_0'' - 2R_3''$
x_2	0	1	0	$\frac{1}{2}$	$\frac{1}{2}$	5	$R_1''' \leftarrow R_1'' + \frac{1}{2}R_3''$
x_1	1	0	0	0	1	3	$R_2''' \leftarrow R_2'' + R_3''$
x_3	0	0	1	-$\frac{1}{2}$	$\frac{3}{2}$	3	$R_3''' \leftarrow \frac{3}{2}R_3''$ (go first)

In the last tableau, there is no positive reduced costs and it is the final optimal tableau.
The optimal bfs for the LP in standard form is

$$(x_1, x_2, x_3, x_4, x_5) \ = \ (3, 5, 3, 0, 0),$$

and the optimal solution for the original LP problem is

$$(x_1, x_2) = (3, 5).$$

The optimal objective value is $z_{\min} = -13$.

1 Further Examples of the Simplex Method

1.1 Example 1 – Unique Optimal Solution

Consider the following minimisation LP.

$$\begin{array}{lrcl}
\min z = & x_1 + x_2 - 4x_3 & & \\
s.t. & x_1 + \ x_2 + 2x_3 + x_4 \qquad\qquad & = & 9 \\
 & x_1 + x_2 - \ x_3 \qquad + x_5 \qquad & = & 2 \\
 & -x_1 + x_2 + \ x_3 \qquad\qquad + x_6 & = & 4 \\
 & x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6 & \geq & 0
\end{array}$$

Note that this LP is already in standard form, and actually in canonical form as well. So we obviously can have an initial basis $\mathbf{x_B} = (x_4, x_5, x_6)^T$.
The first Simplex tableau is

basis	x_1	x_2	x_3	x_4	x_5	x_6	rhs
z	-1	-1	4	0	0	0	0
x_4	1	1	2	1	0	0	9
x_5	1	1	-1	0	1	0	2
x_6	-1	1	$\boxed{1}$	0	0	1	4

The entering variable is x_3, and by the ratio test $\min\{\frac{9}{2}, \frac{4}{1}\}$ we choose x_6 as the leaving variable.[10]
Thus the new basis is $\mathbf{x_B} = (x_4,\ x_5,\ x_3)^T$. Using EROs to make Column 3 become a pivot column gives the second Simplex tableau as below.

basis	x_1	x_2	x_3	x_4	x_5	x_6	rhs	
z	3	-5	0	0	0	-4	-16	$R_0' \leftarrow R_0 - 4R_3$
x_4	$\boxed{3}$	-1	0	1	0	-2	1	$R_1' \leftarrow R_1 - 2R_3$
x_5	0	2	0	0	1	1	6	$R_2' \leftarrow R_2 + R_3$
x_3	-1	1	1	0	0	1	4	

[10]Note that only the basic variables with $\widehat{a}_{it} > 0$ can be considered.

Now x_1 is the only one with a positive reduced cost, so it will enter in the basis. Neither of the last two rows can give us the leaving variable (since they don't have a positive entry in Column 1), so x_4 is the leaving variable. We pivot again and have the following tableau.

basis	x_1	x_2	x_3	x_4	x_5	x_6	rhs	
z	0	-4	0	-1	0	-2	-17	$R_0'' \leftarrow R_0' - R_1'$
x_1	1	-$\frac{1}{3}$	0	$\frac{1}{3}$	0	$-\frac{2}{3}$	$\frac{1}{3}$	$R_1'' \leftarrow \frac{1}{3}R_1'$ (go first)
x_5	0	2	0	0	1	1	6	
x_3	0	$\frac{2}{3}$	1	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{13}{3}$	$R_3'' \leftarrow R_3' + R_1''$

No positive reduced cost exists in row 0, so this is the final optimal tableau. The optimal solution to this LP is $(x_1, x_2, x_3, x_4, x_5, x_6) = (\frac{1}{3}, 0, \frac{13}{3}, 0, 6, 0)$ with the optimal objective value $z_{\min} = -17$.

1.2 Example 2 – Unbounded LP

Consider the following maximisation LP in standard form.

$$\begin{aligned} \max z = \quad & 2x_1 + 3x_2 \\ s.t. \quad & x_1 - x_2 + s_1 = 1 \\ & x_1 - 2x_2 + s_2 = 2 \\ & x_1,\ x_2,\ s_1,\ s_2 \geq 0 \end{aligned}$$

The first Simplex tableau is

basis	x_1	x_2	s_1	s_2	rhs
z	-2	-3	0	0	0
s_1	1	-1	1	0	1
s_2	1	-2	0	1	2

The most negative reduced cost belongs to x_2, so it should enter the basis. However, there is no positive entry $\widehat{a}_{it}$ in its column. A leaving variable cannot be obtained.

This means that as x_2 increases both s_1 and s_2 increase. So neither will become 0. In geometric terms, the feasible region is unbounded along the x_2 axis. Since x_2 has a negative reduced cost, moving in this direction leads to an improvement in the objective value without bound, i.e. the objective tends to ∞. Therefore, this LP problem is unbounded.

The Simplex method can terminate in either of two situations – either finding an optimal solution or having determined the unboundedness of the LP problem.[11]

1.3 Example 3 – Alternative Optimal Solutions

Consider the following minimisation LP in standard form.

$$\begin{array}{rrcl} \min z = & -3x_1 - x_2 - \frac{1}{2}x_3 & & \\ s.t. & 6x_1 \quad\quad - x_3 + s_1 \quad\quad & = & 12 \\ & x_2 \quad + x_3 \quad\quad + s_2 & = & 10 \\ & x_1,\ x_2,\ x_3,\ s_1,\ s_2 & \geq & 0 \end{array}$$

The Simplex algorithm procedure gives the following tableaux

basis	x_1	x_2	x_3	s_1	s_2	rhs
z	3	1	$\frac{1}{2}$	0	0	0
s_1	$\boxed{6}$	0	-1	1	0	12
s_2	0	1	1	0	1	10
z	0	1	1	$-\frac{1}{2}$	0	-6
x_1	1	0	$-\frac{1}{6}$	$\frac{1}{6}$	0	2
s_2	0	$\boxed{1}$	1	0	1	10
z	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	-16
x_1	1	0	$-\frac{1}{6}$	$\frac{1}{6}$	0	2
x_2	0	1	1	0	1	10

[11] In real-world applications, unbounded LPs are usually a sign that the LP models were not properly or correctly formulated.

So an optimal bfs is $(x_1, x_2, x_3, s_1, s_2) = (2, 10, 0, 0, 0)$ with the optimal objective value $z_{\min} = -16$.

Notice that x_3 is a nonbasic variable, but has a reduced cost of 0. This means that it could enter the basis without changing the current objective value which is optimal. Thus, we could have more than one optimal solution, i.e. there could be alternative optimal solutions. Now we proceed to the next Simplex iteration by entering x_3 into the basis as follows.

basis	x_1	x_2	x_3	s_1	s_2	rhs
z	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	-16
x_1	1	0	$-\frac{1}{6}$	$\frac{1}{6}$	0	2
x_2	0	1	$\boxed{1}$	0	1	10
z	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	-16
x_1	1	$\frac{1}{6}$	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{11}{3}$
x_3	0	1	1	0	1	10

So an alternative optimal bfs is $(x_1, x_2, x_3, s_1, s_2) = (\frac{11}{3}, 0, 10, 0, 0)$, undoubtedly with the same optimal objective value $z_{\min} = -16$.

If there is no nonbasic variables with a zero coefficient in row 0 of the optimal tableau, then the LP has a unique optimal solution. Even if there is a nonbasic variable with a zero coefficient in row 0 of the optimal tableau, it is possible that the LP does not have alternative optimal solutions. In solving an LP (at least for this subject), it is sufficient to find "one" optimal solution, if it exists.

1.4 Convergence and Degeneracy of the Simplex Method

On all examples we have seen so far, the Simplex method works. Formally speaking, it converges to an optimal solution, if one exists. But is this actually the case for any LP problem?
The usual argument for why the Simplex method will converge in most cases goes as follows. At each iteration before reaching an optimal solution, we

find a new basis and get a strict improvement in the objective. We can never go back to a previous basis since it will make our objective worse.[12]
Where this argument can fall down is the assumption that we will always get a strict improvement in the objective before obtaining an optimal solution. In some cases, the objective value might stay the same after one iteration. This situation occurring in the Simplex procedure is known as *degeneracy*. It can arise when some of the basic variables have value zero in a bfs.

Consider the following example

$$\begin{array}{rrcl}
\min z = & -\frac{3}{4}x_1 + 150x_2 - \frac{1}{50}x_3 + 6x_4 & & \\
s.t. & \frac{1}{4}x_1 - 60x_2 - \frac{1}{25}x_3 + 9x_4 & \leq & 0 \\
& \frac{1}{2}x_1 - 90x_2 - \frac{1}{50}x_3 + 3x_4 & \leq & 0 \\
& x_3 & \leq & 1 \\
& x_1,\ x_2,\ x_3,\ x_4 & \geq & 0
\end{array}$$

Adding slack variables (x_5, x_6, x_7) respectively to the three constraints, we proceed with the Simplex solution procedure in the usual way. When we encounter a tie in the ratio test, we will select the first row giving the minimum ratio. The initial Simplex tableau is

basis	x_1	x_2	x_3	x_4	x_5	x_6	x_7	rhs
z	$\frac{3}{4}$	-150	$\frac{1}{50}$	-6	0	0	0	0
x_5	$\boxed{\frac{1}{4}}$	-60	$-\frac{1}{25}$	9	1	0	0	0
x_6	$\frac{1}{2}$	-90	$-\frac{1}{50}$	3	0	1	0	0
x_7	0	0	1	0	0	0	1	1

You can notice that in the initial bfs the basic variables x_5 and x_6 both have the value of zero. The result of the first six iterations of the Simplex procedure is shown in the following table.

[12] An individual basic variable can re-enter the basis, but we won't have exactly the same set of basic variables in the basis.

Iteration	$\mathbf{x_B}$	z value
0	$(x_5,\ x_6,\ x_7)$	0
1	$(x_1,\ x_6,\ x_7)$	0
2	$(x_1,\ x_2,\ x_7)$	0
3	$(x_3,\ x_2,\ x_7)$	0
4	$(x_3,\ x_4,\ x_7)$	0
5	$(x_5,\ x_4,\ x_7)$	0
6	$(x_5,\ x_6,\ x_7)$	0

Note that after the first iteration, after the second one, after the third one and so one, the objective value remains unaltered. The degeneracy does occur. Then there may be many bfs that correspond to some non-optimal extreme point. The simplex algorithm might encounter all these sets of bfs before it finds that it was at a non-optimal extreme point. In some cases, the degeneracy just causes more iterations and thus the inefficiency of the Simplex method. The Simplex procedure might still reach an optimal solution even though a degeneracy happens.
Unfortunately, this is not the case for this considered example. Notice that the basis in the 6^{th} iteration is exactly that in the initial Simplex tableau. After all those six iterations, we have gone precisely nowhere. This is known as *cycling*, as the Simplex procedure will return back to some basis again and again. Actually an optimal solution to this LP does exist:

$$(x_1, x_2, x_3, x_4) \;=\; (\frac{2}{50}, 0, 1, 0 \qquad) \quad \text{with} \quad z_{\min} = \frac{-1}{20}.$$

However, the **simple** version of the Simplex method that we have introduced will never find it.
There are a few modifications in the Simplex method to avoid cycling altogether. The most common one is to perturb the rhs slightly, e.g. to change the rhs randomly in this considered LP to

$$\mathbf{b}' = (0.0000001274,\ 0.0000000432,\ 1)^T.$$

From the theoretical perspective, the Simplex procedure will converge to an optimal solution (if one exists) in the absence of degeneracy.

Further reading: Section 4.7–4.8, 4.11 and 6.2 in the reference book "Operations Research: Applications and Algorithms" (Winston, 2004)