Lecture Notes - parts 5-6

Duality of Linear Programming

1 The Normal Form of an LP

An LP in general form satisfying the following form

$$\begin{array}{rrcc} \min\,(\text{or } \max)\; z = & \mathbf{c}^T\mathbf{x} & & \\ s.t. & \mathbf{A}\mathbf{x} & \geq (\text{or } \leq) & \mathbf{b} \\ & \mathbf{x} & \geq & \mathbf{0} \end{array}$$

is said to be in the *normal form*.

Notice in the normal form no sign restriction is imposed on rhs $\mathbf{b}$.[1]

Any LP in general form can be transformed into an equivalent LP in normal form, say minimisation normal form, by the following procedures:

- converting $\max z = \mathbf{c}^T\mathbf{x}$ into $\min z' = -z = -\mathbf{c}^T\mathbf{x}$;
- replacing any urs variable x_i with two nonnegative variables x_i' and x_i'' by setting $x_i = x_i' - x_i''$;
- multiplying any "$\leq$" constraint by -1 to obtain a "$\geq$" constraint;

[1] Recall that nonnegative rhs $\mathbf{b} \geq \mathbf{0}$ is required in the standard form.

- replacing any "=" constraint with a "≥" constraint and a "≤" constraint, e.g. the equality constraint

$$2x_1 + 5x_2 = 7$$

is identical to

$$\begin{aligned} 2x_1 + 5x_2 &\geq 7; \\ 2x_1 + 5x_2 &\leq 7. \end{aligned}$$

Example 1.
Consider the LP

$$\begin{array}{rrcl} \max\ z = & 3x_1 + \ x_2 & & \\ s.t. & 2x_1 + 5x_2 & = & 7 \\ & x_1 + x_2 & \geq & 3 \\ & x_1,\, x_2 & \geq & 0 \end{array}$$

Its minimisation normal form is

$$\begin{array}{rrcr} \min\ z' = & -3x_1 - \ x_2 & & \\ s.t. & 2x_1 + 5x_2 & \geq & 7 \\ & -2x_1 - 5x_2 & \geq & -7 \\ & x_1 + x_2 & \geq & 3 \\ & x_1,\, x_2 & \geq & 0 \end{array}$$

And its maximisation normal form is

$$\begin{array}{rrcr} \max\ z = & 3x_1 + \ x_2 & & \\ s.t. & -2x_1 - 5x_2 & \leq & -7 \\ & 2x_1 + 5x_2 & \leq & 7 \\ & -x_1 - x_2 & \leq & -3 \\ & x_1,\, x_2 & \geq & 0 \end{array}$$

2 The Dual LP

For any LP, there exists a *dual* LP. The dual of an LP in minimisation normal form

$$\begin{array}{rrcll} \min\ z = & \mathbf{c}^T\mathbf{x} & & & \\ s.t. & \mathbf{A}\mathbf{x} & \geq & \mathbf{b} & \qquad \text{(P)} \\ & \mathbf{x} & \geq & \mathbf{0} & \end{array}$$

is the LP

$$\begin{array}{rrcll} \max\ w = & \mathbf{b}^T\mathbf{y} & & & \\ s.t. & \mathbf{A}^T\mathbf{y} & \leq & \mathbf{c} & \qquad \text{(D)} \\ & \mathbf{y} & \geq & \mathbf{0} & \end{array}$$

which is in maximisation normal form. The LP (P) is called the *primal*, and (D) is called the *dual* of (P).
The variable $y_i \in \mathbf{y}$ $(i = 1, 2, \ldots, m)$ is called the *dual variable* associated with the i^{th} constraint of the primal LP.
The variable $x_j \in \mathbf{x}$ $(j = 1, 2, \ldots, n)$ is called the *primal variable* associated with the j^{th} constraint of the dual LP.

The primal (in minimisation normal form) and its dual (in maximisation normal form) can be simultaneously demonstrated in the display

	$\mathbf{x}^T$	
$\mathbf{y}$	$\mathbf{A}$	$\geq \mathbf{b}$
	$\leq \mathbf{c}^T$	

or equivalently

	x_1	x_2	$\cdots$	x_n	
y_1	a_{11}	a_{12}	$\cdots$	a_{1n}	$\geq b_1$
y_2	a_{21}	a_{22}	$\cdots$	a_{2n}	$\geq b_2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
y_m	a_{m1}	a_{m2}	$\cdots$	a_{mn}	$\geq b_m$
	$\leq c_1$	$\leq c_2$	$\cdots$	$\leq c_n$	

Given a primal LP in normal form, we can generate by the above table its dual LP in normal form, which is called the *symmetric* dual. Provided that the given primal LP is not in normal form[2], we can have its *asymmetric* dual by the following rules, which summarise the relationship between constraints and variables in the primal and dual.

primal/dual constraint		**dual/primal variable**
consistent with normal form	$\Longleftrightarrow$	variable ≥ 0
reversed with normal form	$\Longleftrightarrow$	variable ≤ 0
equality constraint	$\Longleftrightarrow$	variable urs

Example 2.

The dual of the LP

$$\begin{array}{rrcl} \min\ z = & 3x_1 + x_2 & & \\ s.t. & 2x_1 + 5x_2 & \geq & 7 \\ & x_1 + 4x_2 & \geq & 2 \\ & x_1,\, x_2 & \geq & 0 \end{array}$$

[2]We could convert it into an equivalent LP in normal form, though.

is

$$\begin{aligned} \max\ w = &\ 7y_1 + 2y_2 \\ s.t. &\ 2y_1 + \ y_2 \le 3 \\ &\ 5y_1 + 4y_2 \le 1 \\ &\ y_1,\, y_2 \ge 0 \end{aligned}$$

Example 3.
Find the dual of the LP

$$\begin{aligned} \min\ z = &\ 4x_1 + 12x_2 + x_3 \\ s.t. &\ -x_1 - 4x_2 + x_3 \le 1 \\ &\ 2x_1 + 2x_2 + x_3 \ge 2 \\ &\ x_1,\, x_2,\, x_3 \ge 0 \end{aligned}$$

We first find the normal form of the considered LP

$$\begin{aligned} \min\ z = &\ 4x_1 + 12x_2 + x_3 \\ s.t. &\ x_1 + 4x_2 - x_3 \ge -1 \\ &\ 2x_1 + 2x_2 + x_3 \ge 2 \\ &\ x_1,\, x_2,\, x_3 \ge 0 \end{aligned}$$

Hence, the dual is

$$\begin{aligned} \max\ w = &\ -y_1 + 2y_2 \\ s.t. &\ y_1 + 2y_2 \le 4 \\ &\ 4y_1 + 2y_2 \le 12 \\ &\ -y_1 + \ y_2 \le 1 \\ &\ y_1,\, y_2 \ge 0 \end{aligned}$$

The Primal-Dual table of Example 3 can be shown as follows:

	x_1	x_2	x_3	
y_1	1	4	-1	≥ -1
y_2	2	2	1	$\geq\ 2$
	≤ 4	≤ 12	≤ 1	

3 The Dual Theorem

3.1 The Dual of the Dual

Lemma 1

The dual of the dual is the primal.

Proof

Assume w.l.o.g. that the primal LP is in minimisation normal form (P). Then the dual is exactly (D).

Obviously, the minimisation normal form of the dual (D) is

$$\begin{array}{rrcl} \min\ u = & -\mathbf{b}^T\mathbf{y} & & \\ s.t. & -\mathbf{A}^T\mathbf{y} & \geq & -\mathbf{c} \\ & \mathbf{y} & \geq & \mathbf{0} \end{array}$$

Then the dual of the above LP is

$$\begin{array}{rrcl} \max\ v = & -\mathbf{c}^T\mathbf{x} & & \\ s.t. & -(\mathbf{A}^T)^T\mathbf{x} & \leq & -\mathbf{b} \\ & \mathbf{x} & \geq & \mathbf{0} \end{array}$$

Since $(\mathbf{A}^T)^T = \mathbf{A}$, the above LP is equivalent to the primal (P). $\square$

3.2 The Weak Duality

Theorem 2

For any feasible solution $\mathbf{x}$ to the primal LP (P) and any feasible solution $\mathbf{y}$ to the dual LP (D), we have

$$\mathbf{c}^T\mathbf{x} \ \geq \ \mathbf{b}^T\mathbf{y},$$

i.e. $z \geq w$.

Proof

From the constraints of the dual (D), we have $\mathbf{A}^T\mathbf{y} \leq \mathbf{c}$. Transposing this inequality gives $\mathbf{y}^T\mathbf{A} \leq \mathbf{c}^T$.
Since $\mathbf{x} \geq \mathbf{0}$, we have

$$\begin{aligned}
\mathbf{c}^T\mathbf{x} &\geq (\mathbf{y}^T\mathbf{A})\mathbf{x} \\
&= \mathbf{y}^T(\mathbf{A}\mathbf{x}) \\
&\geq \mathbf{y}^T\,\mathbf{b} \quad \text{(from the constraints of the primal (P))} \\
&= \mathbf{b}^T\mathbf{y} \quad \text{(transpose of a scalar).}
\end{aligned}$$

□

Corollary 3

1) If the primal LP is unbounded, then the dual LP is infeasible.[3]

2) If the dual LP is unbounded, then the primal LP is infeasible.

[3] The proof by contradiction is trivial. Assume that the primal LP is unbounded and the dual LP is feasible. Hence there exists a dual feasible solution $\mathbf{y}$ (satisfying $\mathbf{A}^T\mathbf{y} \leq \mathbf{c}$) such that the dual objective value $\mathbf{b}^T\mathbf{y}$ is a finite number. If this is the case, then (as per Theorem 2) $\mathbf{c}^T\mathbf{x}$ has a lower bound, which contradicts the unboundedness of the primal LP.

Example 4.
Consider the primal and its dual in Example 2. A feasible solution to the primal is $(x_1, x_2) = (1, 1)$, which gives the objective value $z = 3 \times 1 + 1 = 4$. A feasible solution to the dual is $(y_1, y_2) = (0, 0.25)$, which gives the objective value $w = 7\times 0+2\times 0.25 = 0.5$. Sure enough, the solutions satisfies Theorem 2. Furthermore, we now know that their optimal objective values exist and satisfy the inequality

$$4 \geq z_{\min} \geq w_{\max} \geq 0.5.$$

Actually the "$\geq$" in the middle can be replaced with "$=$" based on a stronger result that we are going to introduce.

3.3 The Strong Duality

Theorem 4

If an optimal solution to the primal LP (P) is obtained, then an optimal solution to its dual LP (D) can readily be obtained. Both optimal objective values are equal, i.e. $z_{\min} = w_{\max}$. In other words, the equality $\mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{y}$ holds if and only if $\mathbf{x}$ is a primal optimal solution and $\mathbf{y}$ is a dual optimal solution.

Proof

If there exists an optimal solution to the primal (P), then there exists a primal bfs which is optimal. Denote this primal optimal bfs by $\mathbf{x}^*$, and assume that $\mathbf{B}$ and $\mathbf{N}$ are the corresponding basic and nonbasic matrices, respectively. Then the primal optimal bfs can be written in the form ${\mathbf{x}^*}^T = ({\mathbf{x}_\mathbf{N}^*}^T \,|\, {\mathbf{x}_\mathbf{B}^*}^T)$.

We first show that a solution

$$\mathbf{y}^* = (\mathbf{c}_{\mathbf{B}}^T \mathbf{B}^{-1})^T$$

is a feasible solution to the dual (D).
Since the primal (P) is a minimisation LP, for the optimal bfs $\mathbf{x}^*$ all the reduced costs of nonbasic variables must be nonpositive, i.e.

$$\widehat{\mathbf{c}}_{\mathbf{N}}^T = \mathbf{c}_B^T \mathbf{B}^{-1}\mathbf{N} - \mathbf{c}_{\mathbf{N}}^T \leq \mathbf{0}^T \quad \Rightarrow \quad \mathbf{c}_B^T \mathbf{B}^{-1}\mathbf{N} \leq \mathbf{c}_{\mathbf{N}}^T. \tag{1}$$

Hence, we have

$$\begin{aligned} \mathbf{A}^T \mathbf{y}^* &= \mathbf{A}^T (\mathbf{c}_{\mathbf{B}}^T \mathbf{B}^{-1})^T \\ &= (\mathbf{c}_{\mathbf{B}}^T \mathbf{B}^{-1} \mathbf{A})^T \\ &= (\mathbf{c}_{\mathbf{B}}^T \mathbf{B}^{-1} (\mathbf{N} \,|\, \mathbf{B}))^T \\ &= (\mathbf{c}_{\mathbf{B}}^T \mathbf{B}^{-1} \mathbf{N} \,|\, \mathbf{c}_{\mathbf{B}}^T \mathbf{B}^{-1} \mathbf{B})^T \\ &= (\mathbf{c}_{\mathbf{B}}^T \mathbf{B}^{-1} \mathbf{N} \,|\, \mathbf{c}_{\mathbf{B}}^T)^T \\ &\overset{(1)}{\leq} (\mathbf{c}_{\mathbf{N}}^T \,|\, \mathbf{c}_{\mathbf{B}}^T)^T \\ &= \mathbf{c}. \end{aligned}$$

So, $\mathbf{y}^*$ is a feasible solution to the dual (D). The corresponding objective value is

$$\begin{aligned} w^* = \mathbf{b}^T \mathbf{y}^* &= \mathbf{b}^T (\mathbf{c}_{\mathbf{B}}^T \mathbf{B}^{-1})^T \\ &= (\mathbf{c}_{\mathbf{B}}^T \mathbf{B}^{-1} \mathbf{b})^T \\ &= (\mathbf{c}_{\mathbf{B}}^T \mathbf{x}_{\mathbf{B}}^*)^T \\ &= \mathbf{c}_{\mathbf{B}}^T \mathbf{x}_{\mathbf{B}}^* \text{ (transpose of a scalar)} \\ &= z_{\min}. \end{aligned}$$

According to Theorem 2 (weak duality), for any dual feasible solution we have

$$w \leq z_{\min} = w^*.$$

Hence $w^* = w_{\max}$ and $\mathbf{y}^*$ is an optimal solution to the dual (D). And $z_{\min} = w_{\max}$ is shown as well. □

Since the dual of the dual LP is the primal LP, the reverse of Theorem 4 follows immediately.

Corollary 5

If an optimal solution to the dual LP (D) is obtained, an optimal solution to its primal LP (P) can be readily obtained and both optimal objective values are equal.

Example 5.
As an illustration for Theorem 4, consider the primal and its dual in Example 2.
The optimal bfs to the primal LP is $(x_1^*, x_2^*) = (0, 1.4)$ with the optimal objective value $z_{\min} = 1.4$.
The optimal bfs to the dual LP is $(y_1^*, y_2^*) = (0.2, 0)$ with the optimal objective value $w_{\max} = 1.4$.

Note: All LP problems satisfy the weak and strong duality properties. The weak duality actually follows from the strong duality. However, we state them separately for many other optimisation problems satisfy the weak duality but not the strong duality. For integer programming, for example, it is possible to find several different types of dual problems which give bounds on the optimal objective value of the primal problem but do not necessarily give the same optimal objective value.

3.4 The Complementary Slackness

An important result about dual variables is that at optimality for the primal and dual LPs, a dual variable will be zero if the corresponding primal constraint is not "active" or "binding", i.e. there is some slack or surplus on that constraint.[4]

Theorem 6

Let $\mathbf{x}$ be a feasible solution to the primal LP (P) and $\mathbf{y}$ be a feasible solution to the dual LP (D). Both solutions $\mathbf{x}$ and $\mathbf{y}$ are optimal to the primal (P) and dual (D), respectively, if and only if they satisfy the following equalities

$$(\mathbf{c}^T - \mathbf{y}^T\mathbf{A})\mathbf{x} = 0 \quad \text{and} \quad \mathbf{y}^T(\mathbf{b} - \mathbf{A}\mathbf{x}) = 0. \tag{2}$$

Proof

For any feasible solution $\mathbf{x}$ to the primal (P) and any feasible solution $\mathbf{y}$ to the dual (D), we have in the proof of Theorem 2

$$\begin{aligned} \mathbf{c}^T\mathbf{x} &\geq (\mathbf{y}^T\mathbf{A})\,\mathbf{x} \\ &= \mathbf{y}^T(\mathbf{A}\mathbf{x}) \\ &\geq \mathbf{y}^T\mathbf{b} \\ &= \mathbf{b}^T\mathbf{y}. \end{aligned} \tag{3}$$

Theorem 4 implies that $\mathbf{x}$ and $\mathbf{y}$ are optimal to the primal (P) and dual (D), respectively, if and only if $\mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{y}$, i.e. the two inequalities in (3) are satisfied at equality. Those two equalities hold if and only if

$$\mathbf{c}^T\mathbf{x} = \mathbf{y}^T\mathbf{A}\mathbf{x} \quad \text{and} \quad \mathbf{y}^T\mathbf{A}\mathbf{x} = \mathbf{y}^T\mathbf{b},$$

[4] The inequality constraint is not satisfied at equality.

which is exactly (2). $\square$

Note: Since

$$\mathbf{c}^T - \mathbf{y}^T\mathbf{A} \geq \mathbf{0}^T \quad \text{and} \quad \mathbf{x} \geq \mathbf{0},$$

the first equation in (2) is equivalent to

$$(c_j - \mathbf{y}^T A_j)x_j = 0,\ \forall j = 1, \ldots, n. \tag{4}$$

This means that for each j we have $x_j = 0$ or $c_j - \mathbf{y}^T A_j = 0$. Similarly, the second equation in (2) is equivalent to

$$y_i(\mathbf{b} - \mathbf{A}\mathbf{x})_i = 0,\ \forall i = 1, \ldots, m. \tag{5}$$

This means that for each i we have $y_i = 0$ or $(\mathbf{b} - \mathbf{A}\mathbf{x})_i = 0$.

Example 6.

Consider again the primal and its dual in Example 2.
Recall that the optimal bfs to the primal is $(x_1^*, x_2^*) = (0, 1.4)$ with $z_{\min} = 1.4$ and the optimal bfs to the dual is $(y_1^*, y_2^*) = (0.2, 0)$ with $w_{\max} = 1.4$.

In the primal LP, the first constraint is binding ($2x_1^* + 5x_2^* = 2\times0 + 5 \times 1.4 = 7$), and the corresponding dual variable is $y_1^* = 0.2 > 0$. The second constraint is not binding ($x_1^* + 4x_2^* = 0 + 4\times1.4 = 5.6 > 2$), and the corresponding dual variable is $y_2^* = 0$. So Eqns. (5) hold.

In the dual LP, the first constraint is not binding ($2y_1^* + y_2^* = 2 \times 0.2 + 0 = 0.4 < 3$), and the corresponding primal variable is $x_1^* = 0$. The second constraint is binding ($5y_1^* + 4y_2^* = 5 \times 0.2 + 4 \times 0 = 1$), and the corresponding primal variable is $x_2^* = 1.4 > 0$. Hence Eqns. (4) hold.

3.5 Interpretation of the Primal-Dual Problems

A good way to interpret the primal-dual pair is to consider a pair of competing businesses as follows.

The Primal LP: Firm A aims to produce 7kg of gold and 2kg of nickel to meet a contract. Each tonne of ore from mine 1 yields 2kg of gold and 1kg of nickel, whilst each tonne of ore from mine 2 yields 5kg of gold and 4kg of nickel. Mining one tonne from mine 1 costs \$300, but costs \$100 from mine 2. The objective of firm A is to minimise the cost of producing enough gold and nickel to meet the contract.

The problem can be summarised in the table shown below.

	Gold (kg/tonne)	Nickel (kg/tonne)	Cost (\$100/tonne)
Mine 1	2	1	3
Mine 2	5	4	1
Demand (kg)	7	2	

Denote by x_i the amount (tonnes) of ore mined from mine i, $i = 1, 2$. This leads to the primal LP shown in Example 2.

$$\begin{aligned} \min\ z = \quad & 3x_1 + x_2 & & \\ s.t. \quad & 2x_1 + 5x_2 & \geq & \ 7 \\ & x_1 + 4x_2 & \geq & \ 2 \\ & x_1,\, x_2 & \geq & \ 0 \end{aligned}$$

The Dual LP: Now suppose that in the market there exists another firm, say B, which sells gold and nickel.[5] Firm B happens to know everything about firm A's operation costs and tries to set the selling prices ($100/kg) y_1 and y_2 of gold and nickel, respectively, for firm A. The objective of firm B is to maximise the revenue from the sale of 7kg of gold plus 2kg of nickel. Firm B knows that firm A will dig out their own minerals if it is cheaper than buying off firm B. To avoid the situation that the selling price is higher than the cost of mining from mine 1 and 2, firm B shall set the price of 2kg of gold plus 1kg of nickel no greater than $300 and that of 5kg of gold plus 4kg of nickel no greater than $100. This leads to the dual LP shown in Example 2.

$$\begin{aligned} \max\ w = \ & 7y_1 + 2y_2 & & \\ s.t. \quad & 2y_1 + \ y_2 & \leq & \ 3 \\ & 5y_1 + 4y_2 & \leq & \ 1 \\ & y_1,\, y_2 & \geq & \ 0 \end{aligned}$$

Hence, the strong duality implies that the sales revenues that firm B can maximise is exactly the same as the mining cost that firm A can minimise.

Now let's see the complementary slackness in action. Because mine 2 is obviously much better than mine 1, in the primal optimal solution we shall have $x_1 = 0$ and thus $x_2 = \max\{\frac{7}{5}, \frac{2}{4}\} = 1.4$.[6] Since the second constraint in the primal is not binding, in the dual optimal solution we have $y_2 = 0$, which means that firm B

[5] We suppose that only firm A and B exist in the market and this is a two-person game.

[6] Hence, the second constraint in the dual is binding.

would charge nothing for nickel. This seems quite counterintuitive, but arises because there is no "rest of the world." If firm A doesn't buy the nickel, firm B can't sell it anywhere else and hence it is of zero value to them.

3.6 Remarks on Primal-Dual Problems

1. Dual Theorem: If an optimal solution exists for either the primal or the dual problem, then an optimal solution exists for the other one and both objective values are the same.

2. An $m \times n$ primal constraint matrix yields an $n \times m$ dual constraint matrix. For each primal (or dual) variable, there is a corresponding dual (or primal) constraint and vice versa.

3. A urs variable in the primal (or dual) gives a corresponding equality constraint in the dual (or primal) and vice versa.

4. If the primal (or dual) is unbounded, the dual (or primal) is infeasible.

5. If the primal (or dual) is infeasible, the dual (or primal) may be either unbounded or infeasible.

6. Given the primal optimal basis, we can get the dual optimal solution $\mathbf{y} = (\mathbf{c}_{\mathbf{B}}^T \mathbf{B}^{-1})^T$, which is an m-dimensional vector called the dual vector.[7] The information about this vector can be obtained from the reduced-cost row (i.e. row 0) of the

[7] The dual vector is exactly the primal Simplex multiplier introduced in the previous chapter.

primal optimal Simplex tableau (in the columns corresponding to the **primal initial basis**).[8] At the primal optimal solution, the values of $\mathbf{c}_{\mathbf{B}}^T\mathbf{B}^{-1}$ is called the *shadow prices*. For $i = 1, \ldots, m$, the i^{th} constraint of the primal is associated with the i^{th} shadow price, which represents the change in the objective function value for a unit change in the rhs of the i^{th} constraint.[9] Note that some of these prices might be zero.

7. For any pair of primal and dual LPs and any pair of the primal and dual feasible solutions, the objective function value of the maximisation problem is always less than or equal to that of the minimisation problem. At the optimal solutions to both problems, the objective value of the maximisation problem will equal that of the minimisation one. This result can be used to estimate the optimal objective value of an intractable optimisation problem by judicious selection of feasible solutions to the primal and dual. Then the range in which the optimal objective value lies can be yielded.

[8] Please refer to page 5 in Lecture Note – Part 5 and page 4 in Lecture Note – Part 6.

[9] $z = \mathbf{c}_{\mathbf{B}}^T\mathbf{B}^{-1}\mathbf{b}$ for the selected basis $\mathbf{x}_{\mathbf{B}}$.

Example 7.
Consider the following LP:

$$\begin{array}{rrcl} \min\ z = & 5x_1 + 2x_2 & & \\ s.t. & x_1 - \ x_2 & \geq & 3 \\ & 2x_1 + 3x_2 & \geq & 5 \\ & x_1,\, x_2 & \geq & 0 \end{array}$$

Its dual LP is

$$\begin{array}{rrcl} \max\ w = & 3y_1 + 5y_2 & & \\ s.t. & y_1 + 2y_2 & \leq & 5 \\ & -y_1 + 3y_2 & \leq & 2 \\ & y_1,\, y_2 & \geq & 0 \end{array}$$

A feasible solution to the primal is $(x_1, x_2) = (3, 0)$ with $z = 15$, and a feasible solution to the dual is $(y_1, y_2) = (3, 1)$ with $w = 14$. This indicates that the optimal value of both objective functions lies between 14 and 15. As this range is narrow, both feasible solutions are near optimal.

Example 8.
Consider the following LP:

$$\begin{array}{rrcl} \max\ z = & x_1 + 2x_2 + 3x_3 - x_4 & & \\ s.t. & x_1 + 2x_2 + 3x_3 & = & 15 \\ & 2x_1 + \ x_2 + 5x_3 & = & 20 \\ & x_1 + 2x_2 + \ x_3 + x_4 & = & 10 \\ & x_1,\, x_2,\, x_3,\, x_4 & \geq & 0 \end{array}$$

This LP currently has no initial basis and could be solved with the two-phase Simplex method by adding the artificial variables

x_5 into the first constraint and x_6 into the second one. Starting with the initial basis (x_5, x_6, x_4), we can obtain the final optimal tableau shown below. Notice that the two columns corresponding to x_5 and x_6 are kept for further purposes although they shall be eliminated after phase-I Simplex procedure.

basis	x_1	x_2	x_3	x_4	x_5	x_6	rhs
z	0	0	0	1	1	0	15
x_2	0	1	0	$\frac{1}{6}$	$\frac{1}{2}$	-$\frac{1}{3}$	$\frac{5}{2}$
x_3	0	0	1	-$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{5}{2}$
x_1	1	0	0	$\frac{7}{6}$	-$\frac{3}{2}$	$\frac{2}{3}$	$\frac{5}{2}$

Now please refer to page 5 in Lecture Note – Part 5 and page 4 in Lecture Note – Part 6. The vector we obtain from the reduced-cost row corresponding to initial basis $\mathbf{x}_{\mathbf{B_0}}^T = (x_5, x_6, x_4)$ is $(1, 0, 1)$, which is $(\mathbf{c}_{\mathbf{B}}^T\mathbf{B}^{-1} - \mathbf{c}_{\mathbf{B_0}}^T)$.[10] The dual optimal solution $(y_1^*, y_2^*, y_3^*) = \mathbf{c}_{\mathbf{B}}^T\mathbf{B}^{-1}$ is therefore $(1, 0, 1) + (0, 0, -1) = (1, 0, 0)$. The optimal objective value of the dual, whose objective function is "min $w = 15y_1 + 20y_2 + 10y_3$", equals the optimal objective value of the primal, equalling 15.

[10]It is $(\mathbf{c}_{\mathbf{B}}^T\mathbf{B}^{-1} - \mathbf{c}_{\mathbf{B_0}}^T)$ rather than $\mathbf{c}_{\mathbf{B}}^T\mathbf{B}^{-1}$ because $\mathbf{c}_{\mathbf{B_0}}^T$ is not a zero vector in the considered LP model.

4 The Dual Simplex Method

The Simplex method we have used (which we could now refer to as the *primal Simplex method*) finds a series of solutions, each of which satisfies the feasibility conditions, until it reaches one which also satisfies the optimality condition. The *dual Simplex method* finds a series of solutions, each of which satisfies the primal optimality condition, until it reaches one which satisfies the primal feasibility conditions. In some situations that finding an initial bfs as usual may not be possible, we need to utilise the dual Simplex method, which maintains dual feasibility (equivalent to primal optimality) while working towards primal feasibility (equivalent to dual optimality).

Hence, the dual Simplex method can also be used to recover primal feasibility in certain circumstances, such as in sensitivity analysis and parametric programming. The implementation of the dual Simplex algorithm is actually simple. We use the existing primal Simplex tableau and apply the entering and leaving rules in reverse order, i.e. work out the leaving variable first and then find the entering variable.

We will consider two types of applications of the dual Simplex method. In the first example, we'll illustrate the algorithm to obtain an initial bfs that could be difficult to obtain using the primal Simplex method. In the second example, we'll use the dual Simplex method to recover the feasibility.

4.1 Generating an Initial bfs

Example 9.

Suppose that we aim to solve the following LP:

$$\begin{aligned} \max\ z = \ & 4x_1 + 5x_2 \\ s.t. \quad & 2x_1 + 3x_2 \leq 6 \\ & 3x_1 + x_2 \geq 3 \\ & x_1,\ x_2 \geq 0. \end{aligned}$$

We first convert the general form to the standard form, giving

$$\begin{aligned} \max\ z = \ & 4x_1 + 5x_2 \\ s.t. \quad & 2x_1 + 3x_2 + s_1 = 6 \\ & 3x_1 + x_2 - s_2 = 3 \\ & x_1,\ x_2,\ s_1,\ s_2 \geq 0. \end{aligned}$$

Since selecting an initial basis of (s_1, s_2) will end up with $s_2 = -3$, which is infeasible, we would not be able to proceed with the Simplex method. Apart from introducing the artificial variables and applying the two-phase Simplex or big-M method, this issue can be tackled by the dual Simplex method (actually much easier). Now we will ignore the infeasibility in setting up the initial Simplex tableau. Thus, the second constraint can be rewritten in the form

$$-3x_1 - x_2 + s_2 = -3.$$

And the initial dual Simplex tableau is shown below.

basis	x_1	x_2	s_1	s_2	rhs
z	-4	-5	0	0	0
s_1	2	3	1	0	6
s_2	-3	-1	0	1	-3

In order to recover primal feasibility, we need to determine the variable that leaves the basis – this is the basic variable that has the most **negative** value. If there is no such variable, then primal feasibility is recovered and the dual Simplex procedure stops.
Since the current basic variable with the most negative value is s_2, it will leave the basis. The entering variable is then determined by the dual ratio test, and the ratio is calculated along the **leaving-variable row**, i.e. the pivot row. The dual ratio is defined as the **absolute value** of the reduced cost divided by the corresponding entry in the pivot row with **negative** value. If there is no entry with negative value in the pivot row, the LP is unbounded in its dual and thus infeasible (in the primal). Otherwise, we choose the variable with the **minimum** ratio as the entering variable. We can append the ratio row to the dual Simplex tableau as follows.

basis	x_1	x_2	s_1	s_2	rhs
z	-4	-5	0	0	0
s_1	2	3	1	0	6
s_2	$\boxed{-3}$	-1	0	1	-3
ratio	$\lvert\frac{-4}{-3}\rvert$	$\lvert\frac{-5}{-1}\rvert$			

Since we have $\min\{|\frac{-4}{-3}|, |\frac{-5}{-1}|\} = |\frac{-4}{-3}|$, the entering variable is x_1. We now do the pivoting on the element at the intersection of the leaving-variable row (pivot row) and the entering-variable column (pivot column) with EROs as shown below.

basis	x_1	x_2	s_1	s_2	rhs	
z	0	$-\frac{11}{3}$	0	$-\frac{4}{3}$	4	$R_0' \leftarrow R_0 + 4R_2'$
s_1	0	$\frac{7}{3}$	1	$\frac{2}{3}$	4	$R_1' \leftarrow R_1 - 2R_2'$
x_1	1	$\frac{1}{3}$	0	$-\frac{1}{3}$	1	$R_2' \leftarrow -\frac{1}{3}R_2$

Notice that the primal feasibility is recovered as all the basic variables are nonnegative. Hence, we terminate the dual Simplex procedure and switch to the primal Simplex procedure as shown below.

basis	x_1	x_2	s_1	s_2	rhs
z	0	$-\frac{11}{3}$	0	$-\frac{4}{3}$	4
s_1	0	$\boxed{\frac{7}{3}}$	1	$\frac{2}{3}$	4
x_1	1	$\frac{1}{3}$	0	$-\frac{1}{3}$	1
z	0	0	$\frac{11}{7}$	$-\frac{2}{7}$	$\frac{72}{7}$
x_2	0	1	$\frac{3}{7}$	$\boxed{\frac{2}{7}}$	$\frac{12}{7}$
x_1	1	0	$-\frac{1}{7}$	$-\frac{3}{7}$	$\frac{3}{7}$
z	0	1	2	0	12
s_2	0	$\frac{7}{2}$	$\frac{3}{2}$	1	6
x_1	1	$\frac{3}{2}$	$\frac{1}{2}$	0	3

Since all the reduced costs are nonnegative, this is the final optimal tableau. The optimal solution is $(x_1, x_2, s_1, s_2) = (3, 0, 0, 6)$ with the optimal objective value $z_{\max} = 12$.

The Dual Simplex Algorithm in Tabular Form

Assume that we have an initial dual Simplex tableau, where some basic variable(s) have negative value(s). Then the following three-step procedure, called the dual Simplex algorithm, will find a bfs for the considered LP in standard form.

Step 1. (Leaving) If $\widehat{\mathbf{b}} = \mathbf{B}^{-1}\mathbf{b} \geq \mathbf{0}$, then STOP – a bfs is found. Otherwise, select the leaving variable x_s whose rhs $\widehat{b}_s$ is the most negative among $\widehat{\mathbf{b}} = \mathbf{B}^{-1}\mathbf{b} = (\widehat{b}_1, \widehat{b}_2, \cdots, \widehat{b}_m)$.

Step 2. (Entering) Label the entries of the row corresponding to the leaving variable x_s by $\widehat{a}_{sj}$, $j = 1, 2, \ldots, n$.
If $\widehat{a}_{sj} \geq 0, \forall j = 1, 2, \ldots, n$, then STOP – the considered LP (the primal) is infeasible.
Otherwise, find

$$t = \arg \min_{1 \leq j \leq n} \left\{ \left| \frac{\widehat{c}_j}{\widehat{a}_{sj}} \right| : \widehat{a}_{sj} < 0 \right\}.$$

Step 3. (Pivoting) Update the tableau by pivoting on $\widehat{a}_{st}$, i.e. perform EROs on the tableau to get a 1 in the pivot position, and 0s above and below it. GO TO Step 1.

4.2 Recovering the Feasibility

If at some iteration of primal Simplex method we arrive at the rhs $\widehat{\mathbf{b}} = \mathbf{B}^{-1}\mathbf{b} \geq \mathbf{0}$, some components of which are zeros, then to avoid the degeneracy the choice of the leaving variable may lead to a violation of feasibility. In this case, the dual Simplex procedure can be applied to recover the feasibility of the basic solution.

Example 10.

Suppose that we aim to solve the following LP:

$$\begin{array}{rrcl} \max\ z = & 0.1x_1 + 0.15x_2 & & \\ s.t. & x_1 + x_2 & \leq & 100000 \\ & -\frac{3}{4}x_1 + \frac{1}{4}x_2 & \leq & 0 \\ & -\frac{3}{2}x_1 + x_2 & \leq & 0 \\ & x_1,\ x_2 & \geq & 0 \end{array}$$

Its standard form is shown as below.

$$\begin{array}{rrcl} \max\ z = & 0.1x_1 + 0.15x_2 & & \\ s.t. & x_1 + x_2 + s_1 & = & 100000 \\ & -\frac{3}{4}x_1 + \frac{1}{4}x_2 + s_2 & = & 0 \\ & -\frac{3}{2}x_1 + x_2 + s_3 & = & 0 \\ & x_1,\ x_2,\ s_1,\ s_2,\ s_3 & \geq & 0 \end{array}$$

An obvious initial basis is $\mathbf{x_B} = (s_1, s_2, s_3)$, and the initial tableau is shown as below.

basis	x_1	x_2	s_1	s_2	s_3	rhs
z	$-\frac{1}{10}$	$-\frac{3}{20}$	0	0	0	0
s_1	1	$\boxed{1}$	1	0	0	100000
s_2	$-\frac{3}{4}$	$\frac{1}{4}$	0	1	0	0
s_3	$-\frac{3}{2}$	1	0	0	1	0

Since x_2 has the most negative reduced cost, it is the entering variable. Notice that the values of the basic variables s_2 and s_3 are 0s. If we choose s_2 or s_3 as the leaving variable, the objective value at the next Simplex tableau will remain the same. Then it will cause the degeneracy and even more terrible cycling. Hence, we turn to try leaving s_1. Then the pivoting results in the following Simplex tableau.

basis	x_1	x_2	s_1	s_2	s_3	rhs
z	$\frac{1}{20}$	0	$\frac{3}{20}$	0	0	15000
x_2	1	1	1	0	0	100000
s_2	-1	0	$-\frac{1}{4}$	1	0	-25000
s_3	$\boxed{-\frac{5}{2}}$	0	-1	0	1	-100000

As both values of the basic variables s_2 and s_3 are negative, the current basic solution is infeasible. Now we need to utilise the dual Simplex algorithm to recover the feasibility. Hence, we select s_3 as the leaving variable, and do the dual ratio test $\min\left\{\left|\frac{\frac{1}{20}}{-\frac{5}{2}}\right|, \left|\frac{\frac{3}{20}}{-1}\right|\right\}$ to determine the entering variable. Thus the entering variable is x_1. Then the pivoting on the pivot element leads to the following tableau.

basis	x_1	x_2	s_1	s_2	s_3	rhs
z	0	0	$\frac{13}{100}$	0	$\frac{1}{50}$	13000
x_2	0	1	$\frac{3}{20}$	0	-$\frac{2}{5}$	15000
s_2	0	0	$\frac{3}{5}$	1	$\frac{2}{5}$	60000
x_1	1	0	$\frac{2}{5}$	0	-$\frac{2}{5}$	40000

Since all the rhs values are nonnegative, we can terminate dual Simplex procedure. As all the reduced costs are nonnegative, this is the final optimal tableau.
Hence, the optimal solution is $(x_1, x_2, s_1, s_2, s_3) =$ (40000, 15000, $0, 60000, 0)$ with the optimal objective value $z_{\text{max}} = 13000$.

Further reading: Section 6.5–6.11 in the reference book "Operations Research: Applications and Algorithms" (Winston, 2004)