37242
Optimisation in Quantitative Management

Preparation Week Study Sheets

This study material is provided to assist students in overcoming any weaknesses in prerequisite knowledge that you might have for the subject 37242 OQM.

1 Revision of Basic Linear Algebra

1.1 Vectors and Matrices

Linear algebra is in some sense the "language" of mathematical optimisation in general. It is a very important shorthand for the problems in higher dimensions that we will encounter.

Vector: A vector of dimension n is an ordered collection of n elements, which are called *components*.

Consider an n-dimensional variable vector

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Vector $\mathbf{x}$ is called non-negative, denoted by $\mathbf{x} \geq \mathbf{0}$, if we have $x_i \geq 0$ for each $i \in \{1, 2, \cdots, n\}$.

In this subject, vectors are written by default as column vectors. When we want a row vector, we write, for example

$$\mathbf{c}^T = (c_1, c_2, \ldots, c_n),$$

which is called $\mathbf{c}$ *transpose*.

If two vectors have the same dimension, then we can take the *dot product* (or *inner product*), which gives us a *scalar*. This can be written in a few ways:

$$\mathbf{c} \cdot \mathbf{x} = \mathbf{c}^T\mathbf{x} = c_1x_1 + c_2x_2 + \ldots + c_nx_n = \sum_{k=1}^{n} c_kx_k.$$

Mostly, the notation $\mathbf{c}^T\mathbf{x}$ will be adopted in this subject.

Matrix: A matrix is an array of numbers, for example,

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

is an $m \times n$ matrix (m rows and n columns).

For each $j \in \{1, 2, \cdots, n\}$, let $\mathbf{A}_j$ be the vector of the j^{th} column of matrix $\mathbf{A}$, i.e.

$$\mathbf{A}_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}.$$

Then matrix $\mathbf{A}$ can be written in the form

$$\mathbf{A} = (\mathbf{A}_1,\ \mathbf{A}_2,\ \cdots\ \mathbf{A}_n).$$

Matrices can also be transposed. The transpose of $\mathbf{A}$ is denoted by $\mathbf{A}^T$, is given by

$$\mathbf{A}^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix},$$

which is an $n \times m$ matrix (n rows and m columns).

A matrix can be multiplied by another one if the number of columns in the first one is identical to the number of rows in the second one. If $\mathbf{A}$ is an

$m \times n$ matrix, and $\mathbf{B}$ is an $n \times p$ matrix, then $\mathbf{AB}$ is an $m \times p$ matrix, given by

$$\mathbf{AB} =$$
$$\begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{n1} & \cdots & a_{11}b_{1p} + a_{12}b_{2p} + \cdots + a_{1n}b_{np} \\ a_{21}b_{11} + a_{22}b_{21} + \cdots + a_{2n}b_{n1} & \cdots & a_{21}b_{1p} + a_{22}b_{2p} + \cdots + a_{2n}b_{np} \\ \vdots & & \vdots \\ a_{m1}b_{11} + a_{m2}b_{21} + \cdots + a_{mn}b_{n1} & \cdots & a_{m1}b_{1p} + a_{m2}b_{2p} + \cdots + a_{mn}b_{np} \end{pmatrix}.$$

Each element of the new matrix, $(\mathbf{AB})_{ij}$, is the dot product of the i^{th} row of $\mathbf{A}$ and the j^{th} column of $\mathbf{B}$. Note that, in general $\mathbf{AB} \neq \mathbf{BA}$.

An n-dimensional (column) vector can be regarded as an $n \times 1$ matrix, and its transpose is a $1 \times n$ matrix. Hence $\mathbf{c}^T\mathbf{x}$ is just a special case of matrix multiplication.

We will also use the product of a matrix and a vector as shown below frequently in this subject.

$$\mathbf{Ax} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix}$$
$$= \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix}$$

One of the ways we use matrices is to abbreviate a whole list of linear inequalities, for example, we will write $\mathbf{Ax} \leq \mathbf{b}$, where $\mathbf{A}$ is an $m \times n$ matrix, $\mathbf{x}$ is an n-dimensional vector and $\mathbf{b}$ is an m-dimensional vector. This is equivalent to

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & \leq & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & \leq & b_2 \\ & \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & \leq & b_m \end{array}$$

We can solve a system of linear equations $\mathbf{Ax} = \mathbf{b}$ by Gaussian elimination (By using elementary row operations, the augmented matrix is reduced to row echelon form). It may have no solution, a unique solution, or an infinite number of solutions.[1]

Consider that $\mathbf{B}$ is a *square matrix* (say $m \times m$). Then its inverse matrix $\mathbf{B}^{-1}$ may exist, and

$$\mathbf{B}\mathbf{B}^{-1} = \mathbf{B}^{-1}\mathbf{B} = \mathbf{I},$$

where $\mathbf{I}$ is an $m \times m$ *identity matrix*.

If $\mathbf{B}^{-1}$ exists, then the system of equations $\mathbf{Bx} = \mathbf{b}$ has a unique solution which can be obtained by

$$\mathbf{x} = \mathbf{B}^{-1}\mathbf{b}.$$

In this case, we can use Gauss-Jordan elimination (By using elementary row operations, the augmented matrix is reduced to reduced row echelon form). To solve $\mathbf{Bx} = \mathbf{b}$ by this method, we set up an *augmented matrix* $(\mathbf{B}|\mathbf{b})$ and use *elementary row operations* (EROs) to reduce the left part of the augmented matrix to the identity matrix.

The three EROs are:

1. Multiply one row by a constant.
2. Add a multiple of one row to another row.
3. Swap two rows.

This takes us from

$$(\mathbf{B}|\mathbf{b})$$

to

$$(\mathbf{B}^{-1}\mathbf{B}|\mathbf{B}^{-1}\mathbf{b}) = (\mathbf{I}|\mathbf{B}^{-1}\mathbf{b}).$$

So the solution is revealed in the right hand side of the resulting augmented matrix.

Consider the following example. To solve the system of equations

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 5, \\ 2x_1 + 4x_2 &= 6, \\ x_1 + 3x_2 &= 6, \end{aligned}$$

[1] Please see p.29–32 in "Operations Research: Applications and Algorithms (Winston, 2004)".

we have the matrix form

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \\ 1 & 3 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ 6 \end{pmatrix}.$$

So its augmented matrix $(\mathbf{B}|\mathbf{b})$ is

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 2 & 4 & 0 & 6 \\ 1 & 3 & 0 & 6 \end{array}\right).$$

Then we conduct the EROs as follows:

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 0 & 0 & -2 & -4 \\ 0 & 1 & -1 & 1 \end{array}\right) \quad \begin{array}{l} R_2' \leftarrow R_2 - 2R_1 \\ R_3' \leftarrow R_3 - R_1 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -2 & -4 \end{array}\right) \quad \begin{array}{l} R_2'' \leftarrow R_3' \\ R_3'' \leftarrow R_2' \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right) \quad R_3''' \leftarrow -\tfrac{1}{2}R_3''$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right) \quad \begin{array}{l} R_1'''' \leftarrow R_1''' - R_3''' \\ R_2'''' \leftarrow R_2''' + R_3''' \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right) \quad R_1''''' \leftarrow R_1'''' - 2R_2''''$$

So the solution is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix}.$$

The matrix inverse can be found by a similar process. Note, though, that finding solutions or finding the inverse can take a substantial amount of work!

After going through this study material, students are urged to read "Chapter 2", including all the examples and exercise problems, in the reference book "Operations Research: Applications and Algorithms (Winston, 2004)."