Introduction to Optimisation:

Linear Programming: Basics. Introduction to Simplex method

Lecture 2

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Standard form

An LP must be presented in the standard form, if we wish to use the *Simplex method*

1. all constraints are in the form of equations
2. all variables are nonnegative
3. all rhs are nonnegative

$$\max\ z\ (\text{or}\ \min z) = cx_1 + c_2x_2 + \cdots + c_nx_n$$

$$\begin{aligned} \textit{s.t.} \qquad & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ & \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \\ & x_1,\ x_2, \ldots, x_n \geq 0 \end{aligned}$$

Nonnegativity of Decision Variables

- Any urs variable x can be presented as $x = p - q$, where $p, q \geq 0$.

- Example:

$$\min z = 2x_1 + 30x_2$$

s.t.

$$4x_1 + 7x_2 \geq 1$$
$$8x_1 + 5x_2 \geq 3 \quad (*)$$
$$6x_1 + 9x_2 \geq -2$$
$$x_1,\ x_2\ urs$$

Set $x_1 = p_1 - q_1$ and $x_2 = p_2 - q_2$ .The equivalent LP (**):

Nonnegativity of Decision Variables

- Show how to construct a solution for the original problem (*) using an optimal solution for the equivalent problem (**) – we assume that it exists.
- Will the constructed solution be optimal for (*)?

Slack and Surplus Variables

- Any inequality constraint can be converted into an equality constraint by adding slack or subtracting surplus nonnegative variables:

$$x_1 - 2x_2 \leq 3 \qquad \Rightarrow \qquad x_1 - 2x_2 + \qquad = 3$$

$$x_1,\ x_2 \geq 0 \qquad \qquad x_1,\ x_2 , \qquad \geq 0$$

$$2x_1 + x_2 \geq 3 \qquad \Rightarrow \qquad 2x_1 + x_2 - \qquad = 3$$

$$x_1,\ x_2 \geq 0 \qquad \qquad x_1,\ x_2, \qquad \geq 0$$

Standard form - summary

To bring an LP to the standard form:

- **Objective function:** if you wish to change objective function from minimisation (or maximisation) form, multiply it by -1 to convert the objective to a maximisation (or minimisation) one.
- **Constraints:** Convert any inequality to an equality constraint by the addition of slack or surplus variables (as appropriate).
- **RHS:** If any rhs b_i is negative, multiply the whole constraint by -1.
- **Variables:** Any $urs\ x_j$ can be replaced by two nonnegative variables x'_j and $x"_j$:

$$x_j = x'_j - x"_j$$

Standard form

The LP problem in the standard form:

$$\max\ z\ (\text{or}\min z) = c^T x$$

$$s.t.\ Ax = b,$$

$$x \geq 0$$

where x and c are n −dimensional vectors, A is an $m \times n$ matrix, and b is an m-dimensional vector. Note that $b \geq 0$.

Standard form - assumptions

- We assume that $(A|b)$ is *consistent* , that is that after application of Gaussian–Jordan method there are no rows $[0\ 0\ 0\ \ldots 0|c]$

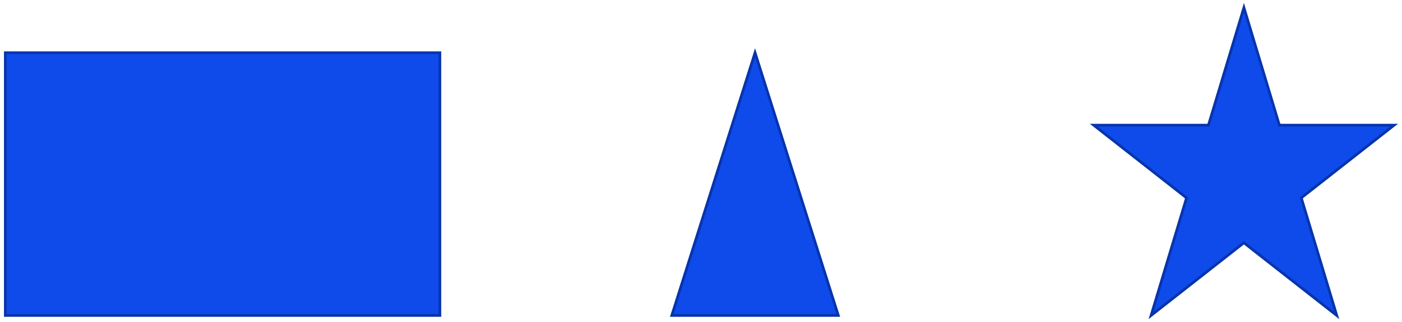
- If $n > m$, then the number of ________, is greater than the number of _________,

Then the system has _________degrees of freedom.

- Give an example of an LP with consistent $(A|b)$ and $n > m$ that's is infeasible (without a feasible solution)

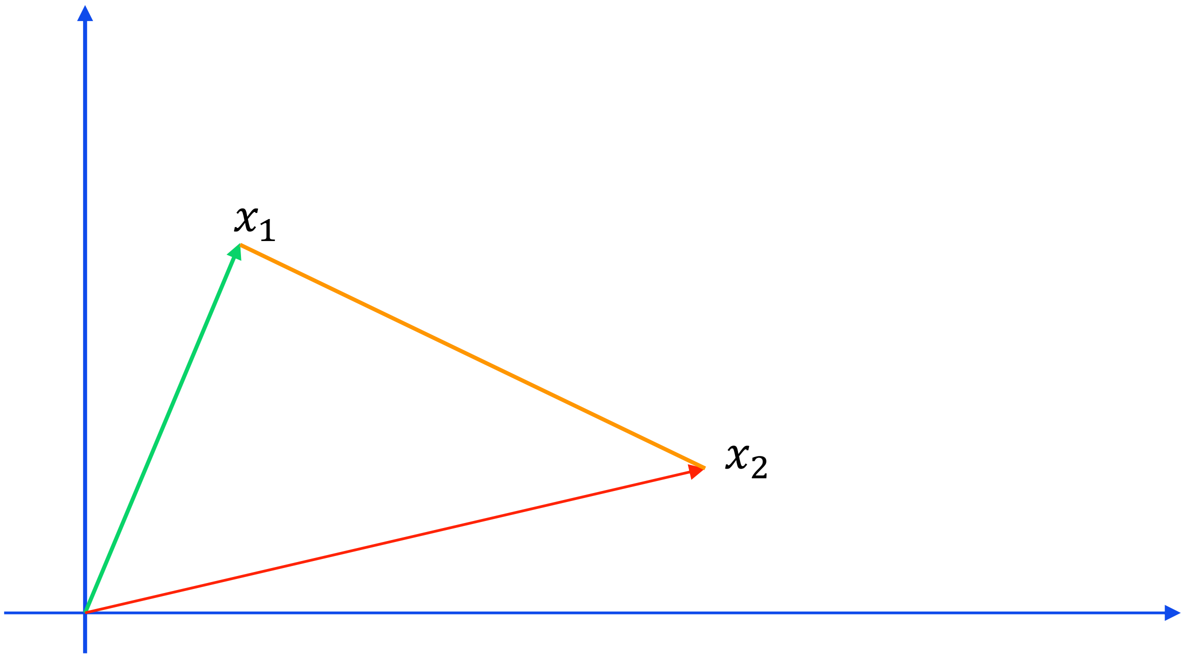
Fundamental Law of LP: definitions

- ***Convex set:*** A set S in n-dimensional space is *convex* if for any two points x_1 and x_2 from S any point of the line segment connecting x_1 and x_2 also belongs to S. In other words, a set S is a convex set if the line segment joining any pair of points in S is wholly contained in S.



Fundamental Law of LP: definitions

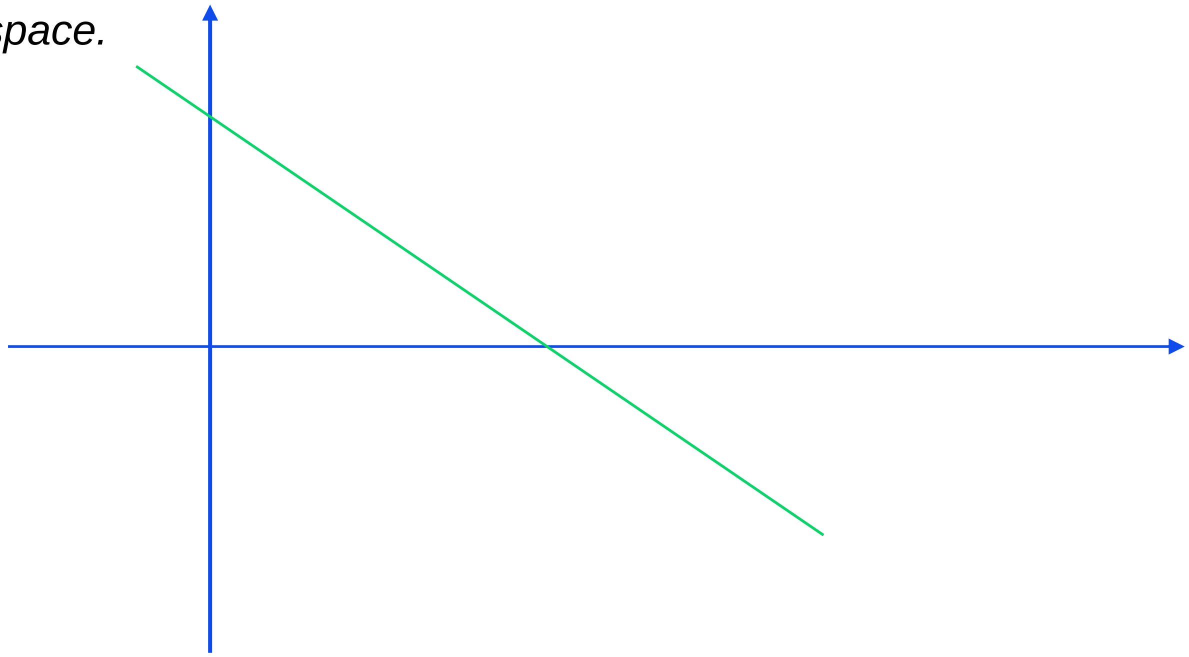
- ***Convex set:*** For any two points x_1 and x_2 in S and $\alpha \in (0,1)$, $x^* = \alpha x_1 + (1 - \alpha)x_2$: $x^* \in S$



Fundamental Law of LP: definitions

- ***Closed half-space:*** for a given an n-dimensional row vector a and a constant b, a $closed\ half - space$ is the set of all vectors (or points) x in n −dimensional space satisfying $ax \leq b$.

- The set of vectors for which $ax = b$ is called ***the boundary*** *of the closed half-space.*



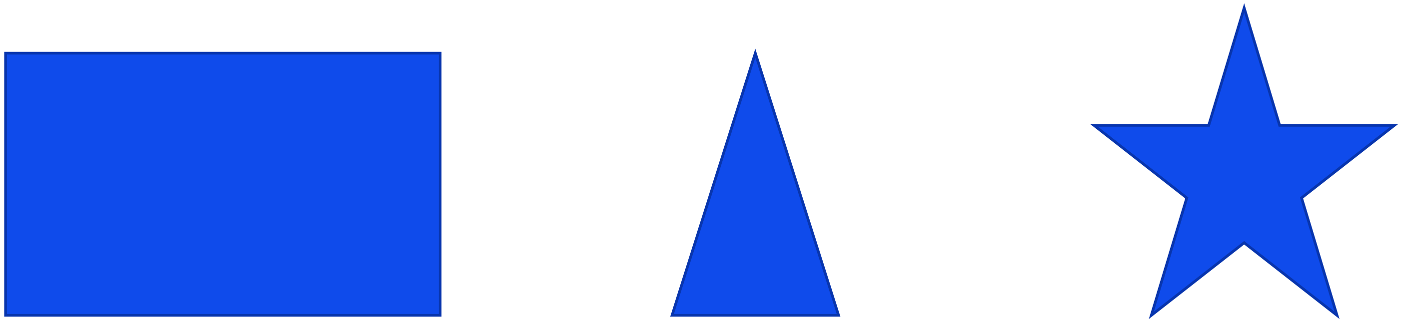
UTS

Fundamental Law of LP: definitions

- ***Extreme point:*** Given a convex set S of n –dimensional vectors, a point x^* is called an *extreme point* (or a corner point) of S if there are no two points x_1 and x_2 in S and a value $\alpha \in (0,1)$, such that

$$x^* = \alpha x_1 + (1 - \alpha) x_2$$

Or any line segment which lies in S and contains x^* has x^* as its end point.



Exercise

- Give an example of convex set with infinite number of extreme points

- Can you find an extreme point for the closed half space which is a convex set?

Fundamental Law of LP: main results

- *Lemma 1 Every closed half-space is a convex set.*

- *Lemma 2* The intersection of any collection of convex sets is a convex set.

Fundamental Law of LP: main results

- ***Theorem 1*** *The feasible set of an LP problem is convex (assuming empty set is convex).*

Fundamental Law of LP: main results

For all results below we assume that an LP in a standard form

- *Lemma 3* If $x = 0$ is a feasible solution of an LP, then it is an extreme point.
- *Lemma 4* For an LP, $x \neq 0$ is an extreme point if and only if the columns of A corresponding to the non-zero x_i are linearly independent.
- *Lemma 5* If the LP is feasible, then it has an extreme point.
- ***Theorem 2*** *The feasible region for any LP has a finite number of extreme points.*
- ***Theorem 3*** If the feasible set is non-empty and one optimal solution exists to the LP, then there is an optimal solution at one of the extreme points.
- For an LP, if objective function value is bounded, then optimal solution exists.

Fundamental Law of LP: main results

Summary of proof:

1. Similar to lemma 4, starting from a feasible solution we can find an extreme point with better objective function value (choose the direction improving the objective function value); otherwise the problem is unbounded.

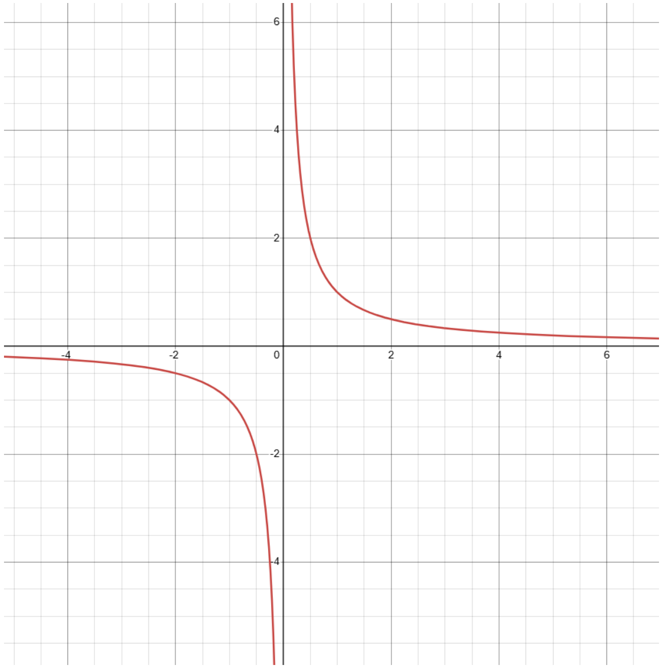
2. If this extreme point is not optimal, we can get a new starting point with better objective function value. We can repeat the previous step to find a better extreme point.

3. As the number of the extreme points are finite, the procedure will terminate with an optimal solution.

Fundamental Law of LP: main results

In general, there may be three cases for the type of optimal objective function value:

- Finite with at least one optimal solution
- Bounded but not obtainable (consider $\min \frac{1}{x} \quad x > 0$)
- Unbounded (therefore no optimal solution, and may/may not have a convergent sequence, consider $\max \frac{1}{x} \quad x > 0$)



Basic Feasible Solution (BFS)

Consider an LP with constraints

$$Ax = b, \quad (1)$$

$$x \geq 0 \quad (2)$$

Assume that $n > m$, $rank(A) = m$, and the feasible region is not empty.

Basic feasible solution:

- Set $n - m$ components of x, to zero.
- Hence if remaining m columns of A are linearly independent, then there exists ______________ solution.
- *Basic* *feasible* *solution is the* ____________ solution for the m components together with_______________zero components.
- The m components are called *basic variables* ________ and the zero components are called *non-basic variables* __________.

Basic Feasible Solution (BFS)

- BFS:

  $(x_1, x_2, x_3, \ldots, x_n) = (\qquad | \qquad)$, and $A = (\qquad | \qquad)$

Then (1) can be presented as :

$$x_B =$$

$$z =$$

Basic Feasible Solution (BFS)

Lemma 6 For an LP, x is an extreme point if and only if it is a basic feasible solution. It is assumed that the LP is in a standard form.

Theorem 4 *If the feasible set is non-empty and one optimal solution exists to the LP, then there is a basic feasible solution giving the optimal value.*

- *Degeneracy* – if more than one bfs represents the same extreme point of the feasible set – to be discussed later…..

Basic Feasible Solution (BFS)

- Example: $max\ 5x_1 + 4x_2$

  s.t. $3x_1 + 2x_2 \leq 120$

  $x_1 + x_2 \leq 50$

  $x_1, x_2 \geq 0$

- Standard form: $max\ 5x_1 + 4x_2 + 0s_1 + 0s_2$

  s.t. $3x_1 + 2x_2 + s_1 = 120$

  $x_1 + x_2 + s_2 = 50$

  $x_1, x_2, s_1, s_2 \geq 0$

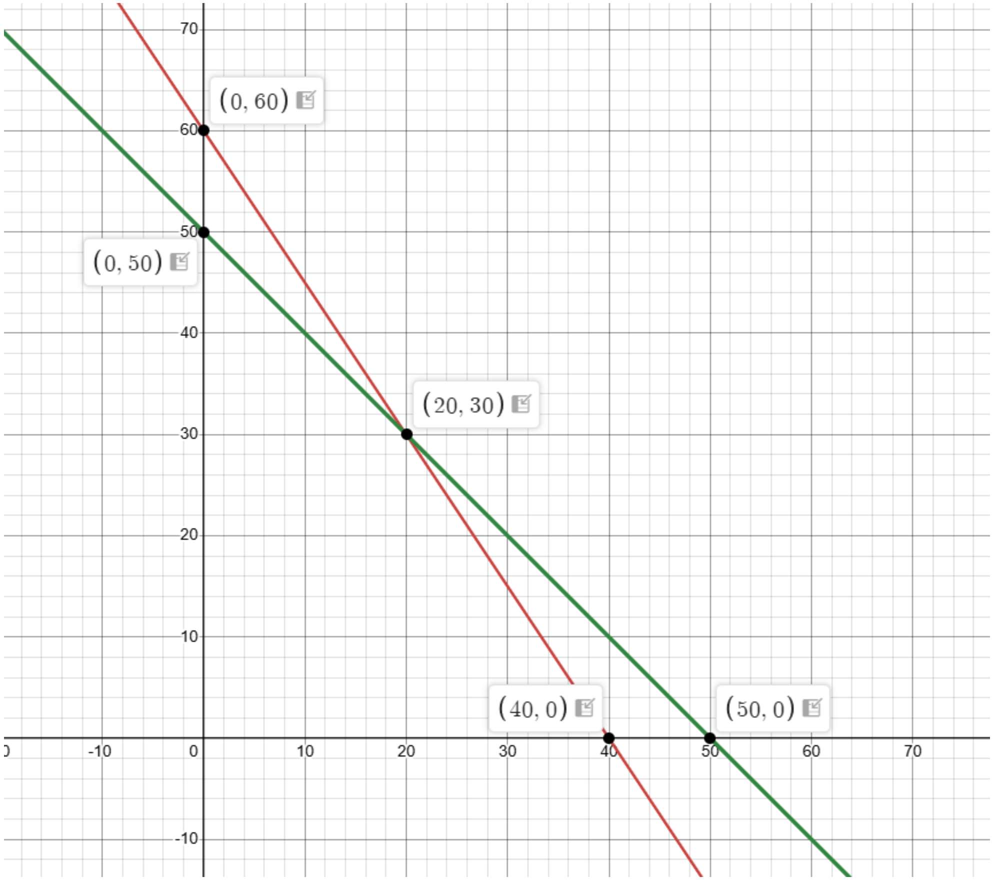
$n =$; $m =$

Basic Feasible Solution (BFS)

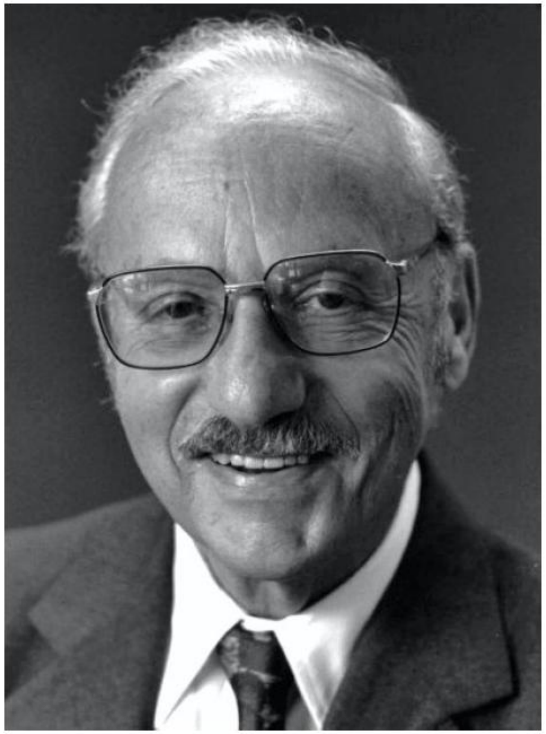
Possible combinations of potential bfs:

Basic Feasible Solution (BFS)

Feasible region:



Simplex method



George Dantzig (1914 - 2005) invented the simplex method in 1947.

In 1954 Dantzig together with Orchard Hays developed the revised simplex algorithm.

Simplex method

- Simplex demonstration
  - https://youtu.be/k9em_7B6298?si=XVchpO-RjaPbMgUf
  - https://youtu.be/k9em_7B6298?si=dgRTTLFmEuFTiacS
- Simplex method performs an efficient search of the extreme points (i.e. bfs) of the feasible region. The method usually starts from the bfs where all original decision variables are zeros.
- Then it "greedily" (in the sense that the objective function value is getting improved) moves from one extreme point (i.e. bfs) of the feasible region to an adjacent bfs by changing one basic variable at a time.
- In the searching/moving procedure, the *ratio test* ensures that the basic solution in each iteration remains feasible (i.e. satisfies all constraints). The method ceases when no further improvement in the value of the objective function
- For any LP with m constraints, two bfs are said to be *"adjacent"* if their bases have $m - 1$ basic variables in common.

Example

- Solve:
$$\begin{aligned} \min z = & -x_1 - 2x_2 \\ s.t. \quad & -2x_1 + x_2 \le 2 \\ & -x_1 + 2x_2 \le 7 \\ & x_1 \le 3 \\ & x_1,\ x_2 \ge 0 \end{aligned}$$

- Standard form:
$$\begin{aligned} \min z = & -x_1 - 2x_2 \\ s.t. \quad & -2x_1 + x_2 + \qquad = 2 \\ & -x_1 + 2x_2 + \qquad = 7 \\ & x_1 + \qquad = 3 \\ & x_1,\ x_2, \qquad \ge 0 \end{aligned}$$

- Each of the constraints has a ________________________ variable.

- $\boldsymbol{x_B} = ($ $\qquad\qquad$ $)$; $\boldsymbol{x_N} = ($ $\qquad\qquad$ $)$;

- Hence $\boldsymbol{bfs}\ \boldsymbol{x} = ($ $\qquad\qquad$ $)$ and the corresponding value of $z =$

Example

- Finding an *adjacent* bfs to improve z:

(What an adjacent bfs?)

1. Express every component of x_B in terms of x_N :

2. Express z in equality form:

3. All OF coefficients for x_N are positive/negative, chose the one with most positive/negative coefficient to enter basis:

To improve z chose ____________, and let _______ $= 0.$

3. To determine the limits of increase for _______ solve:

Hence _______ is entering the basis and _______ is leaving the basis

Example

- New values for components:

New bfs $(x_1, x_2,$ ________ $) = ($ $)$ and $z =$

- New $x_B = ($ $)$; $x_N = ($ $)$

- Express z and every component of x_B in terms of x_N:

- Can we approve z further? Some coefficients for x_N are

Example

- To improve z chose _______, and let _______ $= 0$.

To determine the limits of increase of __________solve:

- New values for components:

- New bfs $(x_1, x_2, x_3, x_4, x_5) = ($ $)$ and $z =$
- New $x_B = ($ $)$; $x_N = ($ $)$;

….and so on…