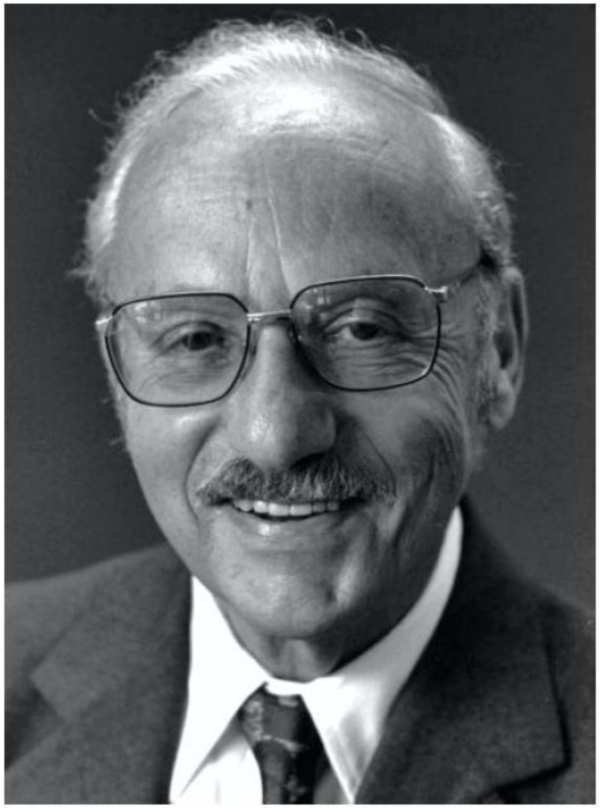
Introduction to Optimisation:

Simplex Method

Lecture 3

Lecture notes by Dr. Julia Memar and Dr. Hanyu Gu and with an acknowledgement to Dr.FJ Hwang and Dr.Van Ha Do

Simplex method



George Dantzig (1914 - 2005) invented the simplex method in 1947.

In 1954 Dantzig together with Orchard Hays developed the revised simplex algorithm.

Simplex method

- Simplex demonstration
  - https://youtu.be/k9em_7B6298?si=XVchpO-RjaPbMgUf
  - https://youtu.be/k9em_7B6298?si=dgRTTLFmEuFTiacS
- Simplex method performs an efficient search of the extreme points (i.e. bfs) of the feasible region. The method usually starts from the bfs where all original decision variables are zeros.
- Then it "greedily" (in the sense that the objective function value is getting improved) moves from one extreme point (i.e. bfs) of the feasible region to an adjacent bfs by changing one basic variable at a time.
- In the searching/moving procedure, the *ratio test* ensures that the basic solution in each iteration remains feasible (i.e. satisfies all constraints). The method ceases when no further improvement in the value of the objective function
- For any LP with m constraints, two bfs are said to be *"adjacent"* if their bases have $m - 1$ basic variables in common.

Example

- Solve:

$$\begin{aligned} \min z = & -x_1 - 2x_2 \\ s.t. \quad & -2x_1 + x_2 \le 2 \\ & -x_1 + 2x_2 \le 7 \\ & x_1 \le 3 \\ & x_1,\ x_2 \ge 0 \end{aligned}$$

- Standard form:

$$\begin{aligned} \min z = & -x_1 - 2x_2 \\ s.t. \quad & -2x_1 + x_2 + \qquad = 2 \\ & -x_1 + 2x_2 + \qquad = 7 \\ & x_1 + \qquad = 3 \\ & x_1,\ x_2, \qquad \ge 0 \end{aligned}$$

- Each of the constraints has a ______________________ variable.

- $\boldsymbol{x_B} = (\qquad\qquad);\quad \boldsymbol{x_N} = (\qquad\qquad);$

- Hence $\boldsymbol{bfs}\ \boldsymbol{x} = (\qquad\qquad)$ and the corresponding value of $z =$

Example

- Finding an *adjacent* bfs to improve z:

(What an adjacent bfs?)

1. Express every component of x_B in terms of x_N :

2. Express z in equality form:

3. All OF coefficients for x_N are positive/negative, chose the one with most positive/negative coefficient to enter basis:

To improve z chose ____________, and let _______ $= 0$.

3. To determine the limits of increase for _______ solve:

Hence ______ is entering the basis and ______ is leaving the basis

Example

- New values for components:

New bfs $(x_1, x_2,$ __________ $) = ($ $)$ and $z =$

- New $x_B = ($ $)$; $x_N = ($ $)$

- Express z and every component of x_B in terms of x_N:

- Can we approve z further? Some coefficients for x_N are

Example

- To improve z chose _______, and let _______$= 0$.

To determine the limits of increase of __________solve:

- New values for components:

- New bfs $(x_1, x_2, x_3, x_4, x_5) = ($ $)$ and $z =$
- New $x_B = ($ $)$; $x_N = ($ $)$;

….and so on…

Simplex Method in a general form

- $\min\ z = c^T x$

  s.t. $\quad Ax = b,$

  $\qquad x \geq 0$

where

$x =$
$c =$
$b =$

$A = [A_1, A_2, \ldots, A_n] =$ $\left(\qquad \right)$

$rank\ A =$

$n \quad m$

Simplex Method in a general form

- Select m variables with linear independent columns in A:

x_B - basis; $B(m \times m)$ - basic matrix;

x_N - non-basis; $N(m \times (n-m))$ -non-basic matrix.

Then

$x^T = [\quad | \quad]$

$c^T = [\quad | \quad]$

$A = [\quad | \quad]$

- Re-write the LP in a *partition form*:

Simplex Method in a general form

- Step 1: current bfs : $x_N =$; hence

- Step 2: to find adjacent bfs formulate z and x_B in terms of x_N:

Reduced cost

Reduced cost of a non-basic variable:

(for the min problem!)

- If all $c_N^T \leq 0$, then z can/cannot be improved

- There is $c_{N_t}^T > 0$: then z can/cannot be improved

Reduced cost

- There is ${c_N^T}_t > 0$: decide what variable is entering/leaving:

1. Select non-basic variable ____ with the largest/smallest $\hat{c_t}$ among all $\widehat{c_N}$_______
2. Ratio Test:
   i. express x_B in terms of x_N:
   ii. Decide what component of x_B is leaving:

Simplex Method in algebraic form summary

- $\min\ (or\ max)\ z = c^T x$

  $s.t. \quad Ax = b,$

  $x \geq 0,$

where $b \geq 0$.

- Obtain the initial bfs;
- Compute the vector $\widehat{c_N^T} =$
  - If $\widehat{c_N^T} \leq 0$ ($\widehat{c_N^T} \geq 0$), then the bfs is ______________
  - Otherwise select x_t with $\widehat{c_{N_t}^T} > 0$ which is the most ____________
    (with $\widehat{c_{N_t}^T} < 0$ which is the most ____________)
- Compute $\widehat{A_t} =$, $\widehat{b_t}$. If $\widehat{A_t} \leq 0$, the LP is ______________
- Otherwise chose $s =$ ______________
- Select $(x_B)_s$ as the leaving component.

Simplex Method in tableau form

- Tableau form:

basis	$\mathbf{x}$	rhs
z	$\mathbf{c}_{\mathbf{B}}^T\mathbf{B}^{-1}\mathbf{A} - \mathbf{c}^T = \widehat{\mathbf{c}}^T$	$\mathbf{c}_{\mathbf{B}}^T\mathbf{B}^{-1}\mathbf{b}$
$\mathbf{x}_{\mathbf{B}}$	$\mathbf{B}^{-1}\mathbf{A} = \widehat{\mathbf{A}}$	$\mathbf{B}^{-1}\mathbf{b} = \widehat{\mathbf{b}}$

- Decomposition of the tableau on x_B and x_N:

basis	$\mathbf{x}_{\mathbf{N}}$	$\mathbf{x}_{\mathbf{B}}$	rhs
z	$\mathbf{c}_{\mathbf{B}}^T\mathbf{B}^{-1}\mathbf{N} - \mathbf{c}_{\mathbf{N}}^T$	$\mathbf{0}^T$	$\mathbf{c}_{\mathbf{B}}^T\mathbf{B}^{-1}\mathbf{b}$
$\mathbf{x}_{\mathbf{B}}$	$\mathbf{B}^{-1}\mathbf{N}$	$\mathbf{I}$	$\mathbf{B}^{-1}\mathbf{b}$

Example

- Solve:

$$\min z = -x_1 - 2x_2$$

s.t.

$$-2x_1 + x_2 \leq 2$$
$$-x_1 + 2x_2 \leq 7$$
$$x_1 \leq 3$$
$$x_1,\ x_2 \geq 0$$

- Standard form:

$$\min z = -x_1 - 2x_2$$

s.t.

$$-2x_1 + x_2 + \qquad = 2$$
$$-x_1 + 2x_2 + \qquad = 7$$
$$x_1 + \qquad = 3$$
$$x_1,\ x_2, \qquad \geq 0$$

Example

Example 2 – unique optimal solution

- Solve: $\min z = x_1 + x_2 - 4x_3$

$$
\begin{aligned}
s.t. \quad & x_1 + x_2 + 2x_3 + x_4 = 9 \\
& x_1 + x_2 - x_3 + x_5 = 2 \\
& -x_1 + x_2 + x_3 + x_6 = 4 \\
& x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6 \geq 0
\end{aligned}
$$

Example 2 – unique optimal solution

Example 3 – unbounded problem

- Standard form:

$$\max z = 2x_1 + 3x_2$$

$$\begin{aligned} s.t. \quad & x_1 - x_2 + s_1 = 1 \\ & x_1 - 2x_2 + s_2 = 2 \\ & x_1,\ x_2, s_1, s_2 \geq 0 \end{aligned}$$

- This problem is ________________________________.

Example 4 – infinite number of optimal solutions

- Standard form:

$$\min z = -3x_1 - x_2 - \frac{1}{2}x_3$$

s.t.

$$6x_1 \quad -x_3 + s_1 \quad = 12$$

$$x_2 \quad + x_3 + \quad + s_2 = 2$$

$$x_1,\ x_2,\ x_3,\ s_1, s_2 \geq 0$$

- Solution in a tableau form:

basis	x_1	x_2	x_3	s_1	s_2	rhs
z	3	1	$\frac{1}{2}$	0	0	0
s_1	$\boxed{6}$	0	-1	1	0	12
s_2	0	1	1	0	1	10
z	0	1	1	$-\frac{1}{2}$	0	-6
x_1	1	0	$-\frac{1}{6}$	$\frac{1}{6}$	0	2
s_2	0	$\boxed{1}$	1	0	1	10
z	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	-16
x_1	1	0	$-\frac{1}{6}$	$\frac{1}{6}$	0	2
x_2	0	1	$\boxed{1}$	0	1	10
z	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	-16
x_1	1	$\frac{1}{6}$	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{11}{3}$
x_3	0	1	1	0	1	10

Convergence and degeneracy of the Simplex Method

- Consider: $\min z = -\frac{3}{4}x_1 + 150x_2 - \frac{1}{50}x_3 + 6x_4$

  s.t.
$$\frac{1}{4}x_1 - 60x_2 - \frac{1}{25}x_3 + 9x_4 + x_5 = 0$$
$$\frac{1}{2}x_1 - 90x_2 - \frac{1}{50}x_3 + 3x_4 + x_6 = 0$$
$$x_3 + x_7 = 1$$
$$x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6,\ x_7 \geq 0$$

- Initial tableau:

basis	x_1	x_2	x_3	x_4	x_5	x_6	x_7	rhs
z	$\frac{3}{4}$	-150	$\frac{1}{50}$	-6	0	0	0	0
x_5	$\boxed{\frac{1}{4}}$	-60	$-\frac{1}{25}$	9	1	0	0	0
x_6	$\frac{1}{2}$	-90	$-\frac{1}{50}$	3	0	1	0	0
x_7	0	0	1	0	0	0	1	1

Convergence and degeneracy of the Simplex Method

- After six iterations:

Cycling!

Iteration	$\mathbf{x_B}$	z value
0	$(x_5,\ x_6,\ x_7)$	0
1	$(x_1,\ x_6,\ x_7)$	0
2	$(x_1,\ x_2,\ x_7)$	0
3	$(x_3,\ x_2,\ x_7)$	0
4	$(x_3,\ x_4,\ x_7)$	0
5	$(x_5,\ x_4,\ x_7)$	0
6	$(x_5,\ x_6,\ x_7)$	0

- To avoid cycling rhs may be changed:

$$b' = (0.0000001274, 0.00000000432, 1)^T$$

Example

- Solve: $\min z = 2x_1 + 3x_2$

$$\begin{aligned} s.t.\quad & \tfrac{1}{2}x_1 + \tfrac{1}{4}x_2 \le 4 \\ & x_1 + 3x_2 \ge 20 \\ & x_1 + x_2 = 10 \\ & x_1,\ x_2 \ge 0 \end{aligned}$$

- Standard form: $\min z = 2x_1 + 3x_2$ original problem (I)

$$\begin{aligned} s.t.\quad & \tfrac{1}{2}x_1 + \tfrac{1}{4}x_2 = 4 \\ & x_1 + 3x_2 = 20 \\ & x_1 + x_2 = 10 \\ & x_1,\ x_2, \ge 0 \end{aligned}$$

- Issues:
  - *The initial solution*
  - *The constraints*

Example – addressing the issues

- Introduce artificial variables:

$$\min z = 2x_1 + 3x_2 \qquad \text{modified problem (II)}$$

$$\text{s.t.} \quad \frac{1}{2}x_1 + \frac{1}{4}x_2 = 4$$

$$x_1 + 3x_2 = 20$$

$$x_1 + x_2 = 10$$

$$x_1,\ x_2, \geq 0$$

- ***Issues again:*** optimal solution for (II) is $(0,0,4,20,10)$. Is it feasible for (I)?

Example – addressing the issues

- Let's introduce "Big $\boldsymbol{M}$":
- $min\, z = 2x_1 + 3x_2 +$ (II)

$$\text{s.t.} \quad \frac{1}{2}x_1 + \frac{1}{4}x_2 = 4$$

$$x_1 + 3x_2 = 20$$

$$x_1 + x_2 = 10$$

$$x_1,\ x_2, \geq 0$$

Big M method

1. Make all $rhs \geq 0$;
2. Add slack/substract excess variables to make equality constraints;
3. For each constraint i *without slack* add an artificial variable a_i;
4. For each a_i add Ma_i for min problems and subtract Ma_i for max problems.
5. Solve the modified problem (II) with Simplex method.

Big M method

- If in optimal solution for (II) all $a_i =$ ___ , then it is _______________for (I)
- If in optimal solution for (II) at least one $a_i >$___, then it is _______________for (I)
- If (II) is unbounded and all $a_i =$ then (I) is _______________________
- If (II) is unbounded and one or more $a_i >$ then (I) is either infeasible or unbounded

Example 1

- $min\, z = 2x_1 + 3x_2$

  s.t. $\frac{1}{2}x_1 + \frac{1}{4}x_2 \leq 4$

  $x_1 + 3x_2 \geq 36$

  $x_1 + x_2 = 10$

  $x_1,\ x_2 \geq 0$

Example 1