Introduction to Optimisation:

Modifications of Simplex Method

Lecture 4

Lecture notes by Dr. Julia Memar and Dr. Hanyu Gu and with an acknowledgement to Dr.FJ Hwang and Dr.Van Ha Do

Example

- Solve:

$$\min z = 2x_1 + 3x_2$$

$$s.t. \quad \frac{1}{2}x_1 + \frac{1}{4}x_2 \le 4$$

$$x_1 + 3x_2 \ge 20$$

$$x_1 + x_2 = 10$$

$$x_1,\ x_2 \ge 0$$

- Standard form: $\min z = 2x_1 + 3x_2$ original problem (I)

$$s.t. \quad \frac{1}{2}x_1 + \frac{1}{4}x_2 + s_1 = 4$$

$$x_1 + 3x_2 - e_2 = 20$$

$$x_1 + x_2 \quad ? \quad = 10$$

NO unique var

$$x_1,\ x_2,\ s_1, e \ge 0$$

- Issues:
  - *The initial solution* $(s_1, e_2) \rightarrow (4, -20) \rightarrow$ NOT feasible
  - *The constraints*

constr. (3) is not satisfied

UTS

Example – addressing the issues

IDEA!

- Introduce artificial variables: to address initial BFS issue

$$\min z = 2x_1 + 3x_2 \qquad \text{modified problem (II)}$$

$$
\begin{aligned}
\text{s.t.} \quad & \tfrac{1}{2}x_1 + \tfrac{1}{4}x_2 + s_1 && = 4 \\
& x_1 + 3x_2 - e_2 + a_2 && = 20 \\
& x_1 + x_2 + a_3 && = 10 \\
& x_1,\ x_2,\ s_1\ e_2\ a_2\ a_3 && \geq 0
\end{aligned}
$$

- ***Issues again:*** optimal solution for (II) is ~~[illegible]~~. Is it feasible for (I)?

$x_B = (s_1\ \ a_2\ \ a_3)$ BFS $= (0, 0, 4, 0, 20, 10)$

constr. (3) is not satisfied

Example – addressing the issues

- Let's introduce "Big $\boldsymbol{M}$":
- $min\, z = 2x_1 + 3x_2 + Ma_2 + Ma_3$ (II)

$$
\text{s.t.} \quad
\begin{aligned}
\tfrac{1}{2}x_1 + \tfrac{1}{4}x_2 + s_1 &= 4 \\
x_1 + 3x_2 - e_2 + a_2 &= 20 \\
x_1 + x_2 + a_3 &= 10 \\
x_1,\ x_2,\ s_1\ e_2\ a_2\ a_3 &\geq 0
\end{aligned}
$$

Big M method

1. Make all $rhs \geq 0$;
2. Add slack/substract excess variables to make equality constraints;
3. For each constraint i *without slack* add an artificial variable a_i;
4. For each a_i add Ma_i for min problems and subtract Ma_i for max problems.
5. Solve the modified problem (II) with Simplex method.

Big M method

- If in optimal solution for (II) all $a_i = \underline{\emptyset}$, then it is <u>optimal</u> for (I)
- If in optimal solution for (II) at least one $a_i > \underline{0}$, then it is <u>Infeas.</u> for (I) and (I) Infeasible
- If (II) is unbounded and all $a_i = 0$ then (I) is <u>Infeas/unbounded</u>
- If (II) is unbounded and one or more $a_i > 0$ then (I) is either infeasible or unbounded

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Example 2 – solve with Big M method

- Let's introduce "Big M":
  - $min\, z = 2x_1 + 3x_2 + Ma_2 + Ma_3$ (II)

s.t.

$$\frac{1}{2}x_1 + \frac{1}{4}x_2 + s_1 = 4$$

$$x_1 + 3x_2 - e_2 + a_2 = 20$$

$$x_1 + x_2 + a_3 = 10$$

$$x_1,\ x_2,\ s_1\ e_2\ a_2\ a_3 \geq 0$$

$x_{B_0} = (s_1\ a_2\ a_3)$ $x_{N_0} = (x_1\ x_2\ e_2)$

Linear form OF :

$$z - 2x_1 - 3x_2 - Ma_2 - Ma_3 = 0$$

canonical form

basis	$\mathbf{x_N}$	$\mathbf{x_B}$	rhs
z	$\underbrace{\mathbf{c_B^T B^{-1} N - c_N^T}}_{\bar{c}_N}$	$\underbrace{\mathbf{0}^T}_{\bar{c}_B}$	$\mathbf{c_B^T B^{-1} b}$
$\mathbf{x_B}$	$\mathbf{B^{-1} N}$	$\mathbf{I}$	$\mathbf{B^{-1} b}$

Initial tableau is not in canonical form

$+ MR_2 + MR_3$

Example 2 – solve with Big M method

	x_1	x_2	s_1	e_2	a_2	a_3	RHS
z	-2	-3	0	0	$-M$	$-M$	0
s_1	$1/2$	$1/4$	1	0	0	0	4
a_2	1	3	0	-1	1	0	20
a_3	1	1	0	0	0	1	10

	x_1	x_2	s_1	e_2	a_2	a_3	RHS	
z	$2M-2$	$\boxed{4M-3}$	0	$-M$	0	0	$30M$	
s_1	$1/2$	$1/4$	1	0	0	0	4	$4 \div \frac{1}{4}$
a_2	1	$\boxed{3}$	0	-1	1	0	20	$20 \div 3$
a_3	1	1	0	0	0	1	10	$10 \div 1$

z	$\frac{2M}{3}-1$	0	0	$\frac{M}{3}-1$	$-\frac{(4M-3)}{3}$	0	$\frac{10M}{3}+20$
s_1	$5/12$	0	1	$1/12$	$-1/12$	0	$4-\frac{5}{3}=7/3$
x_2	$1/3$	1	0	$-1/3$	$1/3$	0	$\frac{20}{3}$
a_3	$2/3$	0	0	$1/3$	$-1/3$	1	$\frac{10}{3}$
z	0	0	0	$-1/2$	$-M+1/2$	$-3/2\left(\frac{2M}{3}-1\right)$	25
s_1	0	0	1	$-1/8$	$1/8$	$-5/8$	$1/4$
x_2	0	1	0	$-1/2$	$1/2$	$-1/2$	$\frac{15}{3}=5$
x_1	1	0	0	$1/2$	$-1/2$	$3/2$	5

$R_0' = R_0 - \left(\frac{2M}{3}-1\right)R_3'$ $\qquad R_3' = R_3 \times \frac{3}{2}$

$s_1 - 1/8\, e_2 + 1/8\, a_2 - \frac{5}{8}a_3 = 1/4$ $\qquad R_2' = R_2 - R_3/2$

$R_1' = R_1 - \frac{5}{12}R_3'$

$$\frac{M}{3} - 1 - \left(\frac{2M}{3}-1\right)\times\frac{1}{2} = \frac{M}{3} - 1 - \frac{M}{3} + 1/2 = -1/2$$

$$-\frac{(4M-3)}{3} + \left(\frac{2M}{3}-1\right)\times\frac{1}{2} = -\frac{4M}{3} + 1 + \frac{M}{3} - 1/2$$

$$= -M + 1/2$$

$$\frac{10M}{3} + 20 - 5\left(\frac{2M}{3}-1\right) = \frac{10M}{3} + 20 - \frac{10M}{3} + 5 =$$

$$= 25$$

$z^* = 25$ $\qquad$ both a_2 and a_3 are zero

$x_1 = 5$

$x_2 = 5$

$$R_0' = R_0 - (4M-3)R_2'$$

$$2M - 2 - \frac{(4M-3)}{3} = 2M - \frac{4M}{3} - 2 + 1 =$$

$$-M + \frac{(4M-3)}{3} = \frac{M}{3} - 1$$

$$30M - \frac{(4M-3)\times 20}{3} =$$

$$= \frac{90M}{3} - \frac{80M}{3} + 20 = \frac{10M}{3} + 20$$

Ratio test:

$$\frac{7}{3} \div \frac{5}{12} = \frac{7}{\cancel{3}} \times \frac{\cancel{12}^{\,4}}{5} = \frac{28}{5}$$

$$\frac{20}{3} \div \frac{1}{3} = 20$$

$$\frac{10}{3} \div \frac{2}{3} = 5$$

➢ $min\ z = 2x_1 + 3x_2$

$$\text{s.t.}\quad \frac{1}{2}x_1 + \frac{1}{4}x_2 \le 4$$
$$x_1 + 3x_2 \ge 36$$
$$x_1 + x_2 = 10$$
$$x_1,\ x_2 \ge 0$$

$$\min z = 2x_1 + 3x_2 + Ma_2 + Ma_3$$

$$\text{s.t.}\quad \frac{1}{2}x_1 + \frac{1}{4}x_2 + s_1 = 4$$
$$M \quad x_1 + 3x_2 - e_2 + a_2 = 36$$
$$M \quad x_1 + x_2 + a_3 = 10$$
$$x_1\ x_2\ a_2\ a_3\ e_2\ s_1 \ge 0$$

$$x_{B_0} = (s_1\ a_2\ a_3)$$

NOT canonical, as $\hat{c}_{a_2} = \hat{c}_{a_3} = -M$

$$z - 2x_1 - 3x_2 - Ma_2 - Ma_3 = 0$$

$+MR_2 + MR_3$

$$z + (2M-2)x_1 + (4M-3)x_2 - Me_2 = 46M$$

	x_1	x_2	s_1	e_2	a_2	a_3	RHS	
z	$2M-2$	$\boxed{4M-3}$	0	$-M$	0	0	$46M$	
s_1	$1/2$	$1/4$	1	0	0	0	4	16
a_2	1	3	0	-1	1	0	36	12
a_3	1	1	0	0	0	1	10	10
z	$-2M+1$	0	0	$-M$	0	$-(4M-3)$	$6M+30$	
s_1	$1/4$	0	1	0	0	$-1/4$	$3/2$	
a_2	-2	0	0	-1	1	-3	6	
x_2	1	1	0	0	0	1	10	

$$R_3' = R_3$$
$$R_2' = R_2 - 3R_3$$
$$R_1' = R_1 - \frac{1}{4}R_3$$
$$R_0' = R_0 - (4M-3)R_3$$

$$46M - (4M-3)10 = 6M + 30$$

Opt. value for (II) is $6M+30$

and $a_2 = 6 \neq 0 \rightarrow$ (I) Inf/unbounded

Two-phase Simplex method

Phase I .

1. Make sure all $rhs \geq 0$;
2. Add slack/subtract excess variables to make equality constraints;
3. For each constraint i without a slack add artificial variable a_i;
4. Solve $\min w = \sum a_i$ with the modified constraints from step 3.

Two-phase Simplex method

$\min w = \sum a_i$

Phase II

- If $w_{opt} > 0$, then (I) is Infeasible, as at least one $a_i > 0$

- If $w_{opt} = 0$ and there are no a_i in x_B then
  1. use final tableau from Ph. 1
  2. replace R_0 with original OF
  3. drop columns with a_i → solve (I)

- If $w_{opt} = 0$ and there are a_i in x_B then
  1. use final tableau from Ph. 1
  2. replace R_0 with original OF
  3. drop columns with non-basic a_i → solve (I)

Example 3 (Solve with two-phase Simplex)

- $\min z = 2x_1 + 3x_2$

$$s.t. \quad \frac{1}{2}x_1 + \frac{1}{4}x_2 \leq 4$$

$$x_1 + 3x_2 \geq 20$$

$$x_1 + x_2 = 10$$

$$x_1,\ x_2 \geq 0$$

Standard form:

$$\min z = 2x_1 + 3x_2$$

$$s.t. \quad (*) \begin{cases} \frac{1}{2}x_1 + \frac{1}{4}x_2 + s_1 = 4 \\ x_1 + 3x_2 - e_2 + a_2 = 20 \\ x_1 + x_2 + a_3 = 10 \end{cases}$$

$$x_1\ x_2\ s_1\ e_2\ a_2\ a_3 \geq 0$$

Example 3

Phase 1

$$\min \quad w = a_2 + a_3$$

$$\text{s.t.} \quad (*)$$

$$\begin{cases} \frac{1}{2}x_1 + \frac{1}{4}x_2 + s_1 = 4 \\ \cdot x_1 + \cdot 3x_2 - e_2 + a_2 = 20 \\ \cdot x_1 + \cdot x_2 + a_3 = 10 \\ x_1\ x_2\ s_1\ e_2\ a_2\ a_3 \geq 0 \end{cases}$$

$$x_{B_0} = (s_1\ a_2\ a_3)$$

$w - a_2 - a_3 = 0 \rightarrow$ not canonical, as a_2 and a_3 are in x_{B_0}

$+R_2 + R_3$

$$w + 2x_1 + 4x_2 - e_2 = 30$$

	x_1	x_2	s_1	e_2	a_2	a_3	RHS	
w	2	4	0	-1	0	0	30	
s_1	$1/2$	$1/4$	1	0	0	0	4	16
a_2	1	3	0	-1	1	0	20	* $20/3$
a_3	1	1	0	0	0	1	10	10
w	$\boxed{2/3}$	0	0	$1/3$	$-4/3$	0	$\frac{10}{3}$	$R_0' = R_0 - 4R_2$
s_1	$5/12$	0	1	$1/12$	$-1/12$	0	$7/3$	$\frac{28}{5}$
x_2	$1/3$	1	0	$-1/3$	$1/3$	0	$\frac{20}{3}$	20
a_3	$2/3$	0	0	$1/3$	$-1/3$	1	$\frac{10}{3}$	* 5
	$R_0 - R_3^{old}$						see ratio above	
w	0	0	0	0	-1	-1	0	
s_1	0	0	1	$-1/8$	$1/8$	$-5/8$	$1/4$	
x_2	0	1	0	$-1/2$	$1/2$	$-1/2$	5	
x_1	1	0	0	$1/2$	$-1/2$	$3/2$	5	

Phase 1 is solved to optimality

$w^* = 0 \rightarrow$ proceed to Phase 2

$z - 2x_1 - 3x_2 = 0 \leftarrow$ not canonical as x_1 and x_2 in the basis

$R_0 + 3R_2 + 2R_3$

	x_1	x_2	s_1	e_2	a_2	a_3	RHS
z	-2	-3	0	0			0
s_1	0	0	1	$-1/8$	$1/8$	$-5/8$	$1/4$
x_2	0	1	0	$-1/2$	$1/2$	$-1/2$	5
x_1	1	0	0	$1/2$	$-1/2$	$3/2$	5
z	0	0	0	$-3/2+1$			25
s_1	0	0	1	$-1/8$	$1/8$	$-5/8$	$1/4$
x_2	0	1	0	$-1/2$	$1/2$	$-1/2$	5
x_1	1	0	0	$1/2$	$-1/2$	$3/2$	5

optimal as $\hat{c}_N < 0$;

ANS: $z^* = 25$

$x_1^* = x_2^* = 5$

Example 4 (Solve with two-phase Simplex)

Home or tute

- $\max z = 4x_1 + 5x_2$

$$s.t. \quad 2x_1 + 3x_2 \leq 6$$

$$3x_1 + x_2 \geq 3$$

$$x_1, \; x_2 \geq 0$$

Example 4

Revised Simplex method: Introduction

- Large-scale LP problems contain large amounts of data, and
- Require even larger numbers of computations in each Simplex iteration
- Consider $\min\ (\boldsymbol{or\ max})z = \boldsymbol{c}^T\boldsymbol{x}$

$$s.t. \quad \boldsymbol{Ax} = \boldsymbol{b},$$
$$\boldsymbol{x} \geq \boldsymbol{0},$$

For the chosen $\boldsymbol{x_B}$, A, B, c^T, c_B^T can be obtained by observation

After B^{-1} is calculated, we can calculate the tableau:

basis	$\mathbf{x}$	rhs
z	$\mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{A} - \mathbf{c}^T$ $= \widehat{\mathbf{c}}^T$	$\mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{b}$
$\mathbf{x_B}$	$\mathbf{B}^{-1}\mathbf{A}$ $= \widehat{\mathbf{A}}$	$\mathbf{B}^{-1}\mathbf{b}$ $= \widehat{\mathbf{b}}$

Revised Simplex method: Introduction

- Decomposed tableau for $x^T = (x_N | x_B)$:

basis	$\mathbf{x_N}$	$\mathbf{x_B}$	rhs
z	$\mathbf{c_B^T B^{-1} N - c_N^T}$	$\mathbf{0}^T$	$\mathbf{c_B^T B^{-1} b}$
$\mathbf{x_B}$	$\mathbf{B^{-1} N}$	$\mathbf{I}$	$\mathbf{B^{-1} b}$

- For initial x_{B_0} $B_0 =$ I and $A = (N_0 | I)$ and $B_0^{-1} =$ I
- The initial tableau :

usually $\emptyset$

basis	x_{N_0}	x_{B_0}	rhs
z	$c_{B_0}^T B_0^{-1} N_0 - c_{N_0}^T$	0^T	$c_{B_0}^T B_0^{-1} b = \emptyset$
x_{B_0}	N_0	I	$B_0^{-1} b = b$

Revised Simplex method: Introduction

if x_{B_0} is not optimal $\rightarrow$ choose x_{B_1}

For the next x_{B_1}

- $B_1^{-1}(\underbrace{N_0|I}_{A}) = \left(B_1^{-1}N_0 \mid B_1^{-1}\right)$ $\rightarrow$ bringing Tableau for x_{B_1} to canonical form

- $\underbrace{c_{B_1}^T B_1^{-1} A - c^T}_{\text{re-calculate reduced cost for } B_1} = c_{B_1}^T B_1^{-1}(N_0 | I) - (c_{N_0}^T \mid c_{B_0}^T) =$
$= \left(c_{B_1}^T B_1^{-1} N_0 - c_{N_0}^T \mid c_{B_1}^T B_1^{-1} - c_{B_0}^T\right)$

basis	x_{N_0}	x_{B_0}	rhs
z	$c_{B_1}^T B_1^{-1} N_0 - c_{N_0}^T$	$c_{B_1}^T B^{-1} - c_{B_0}^T$	$c_{B_1}^T B^{-1} b$
x_{B_1}	$B_1^{-1} N_0$	B_1^{-1}	$B_1^{-1} b$

Revised Simplex method: Introduction

- Observations:

1. B^{-1} for current basis can be observed in the columns corresponding to x_{B_0}

2. $c_{B_i}^T B_i^{-1} - \underbrace{c_{B_0}^T}_{\substack{\text{usually} \\ = \phi}} \rightarrow$ usefu

Revised Simplex method: Introduction

if x_{B_1} not optimal

If $(x_{N_1}|x_{B_1})$ is not optimal, select the next x_{B_2}

- $B_2^{-1}(N_0|I) = \left(B_2^{-1}N_0 \mid B_2^{-1}\right)$

  $B_2^{-1}(A)$

- $c_{B_2}^T B_2^{-1} A - c^T = \underbrace{c_{B_2}^T B_2^{-1}}\left(N_0 \mid I\right) - \left(c_{N_0}^T \mid c_{B_0}^T\right) =$

basis	x_{N_0}	x_{B_0}	rhs
z	$c_{B_2}^T B_2^{-1} N_0 - c_{N_0}^T$	$c_{B_2}^T B_2^{-1} - \underbrace{c_{B_0}^T}_{=0}$	$c_{B_2}^T B_2^{-1} b$
x_{B_2}	$B^{-1} N_0$	B_2^{-1}	$B_2^{-1}b$