Introduction to Optimisation:

Sensitivity analysis

Lecture 7

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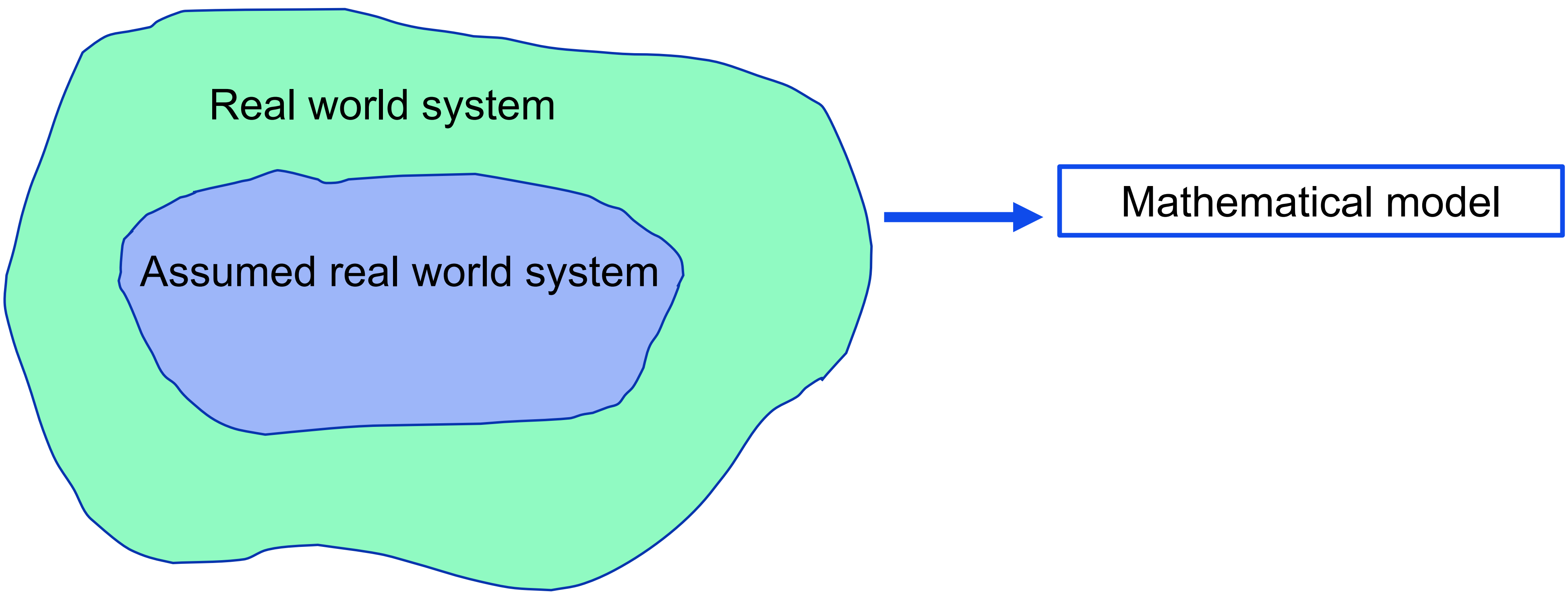
Introduction

Mathematical Optimization modelling - some assumptions:

- Linearity of constraints and functions
- Data certainty
- Values of parameters



Introduction

Sensitivity analysis is a systematic study of how sensitive the LP's optimal solution is to (small) changes in the LP's parameters and it is presented to give answers to questions of the following forms:

$OF: z = \underline{c}^T x$

1. If the objective function changes, how does the optimal solution change?

A RHS

2. If the amount of resources available changes, how does the optimal solution change?

3. If an additional constraint is added to the LP, how does the optimal solution change?

Sensitivity analysis may allow to avoid re-solving an LP if the change in parameters does not imply the change in optimal basis.

Introduction

- Consider $\min\ (\boldsymbol{or\ max})\ z = \boldsymbol{c}^T\boldsymbol{x}$

$$s.t. \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},$$
$$\boldsymbol{x} \geq \boldsymbol{0},$$

where $\boldsymbol{b} \geq \boldsymbol{0}$, and optimal bfs $|\boldsymbol{x}^{*T} = (\boldsymbol{x}_N^T|\boldsymbol{x}_B^T)$ and the optimal tableau: $\downarrow$

basis	$\mathbf{x_N}$	$\mathbf{x_B}$	rhs
z^*	$\overbrace{\mathbf{c_B^T B^{-1} N - c_N^T}}^{\hat{c}_N^T}$	$\mathbf{0}^T$	$\mathbf{c_B^T B^{-1} b}$
$\mathbf{x_B}$	$\mathbf{B^{-1} N}$	$\mathbf{I}$	$\mathbf{B^{-1} b}$

OR

basis	$\mathbf{x_{N_0}}$	$\mathbf{x_{B_0}}$	rhs
z^*	$\mathbf{c_B^T B^{-1} N_0 - c_{N_0}^T}$	$\mathbf{c_B^T B^{-1} - c_{B_0}^T}$	$\mathbf{c_B^T B^{-1} b}$
$\mathbf{x_B}$	$\mathbf{B^{-1} N_0}$	$\mathbf{B^{-1}}$	$\mathbf{B^{-1} b}$

- From the tableau:

$\hat{c}_N^T = c_B^T B^{-1} N - c_N^T \underset{\text{min}}{\leq} (or \underset{\text{max}}{\geq}) 0^T;$ (1) *optimality condition*

- $x_B = B^{-1} b \geq 0;$ (2)
- $x_N = 0;$ (3)

(2)–(3) *feasibility conditions*

- $z^* = c_B^T B^{-1} b$ (4)

Change in the objective function coefficient

Non-basic variable:

- Let x_j be non-basic variable in optimal *bfs*, and change $c_j, j \in N$:

$$c_j' = c_j + \Delta.$$

- How will it affect the solution?

Feasibility? NO
value OF? NO
optimality? yes

$$\hat{c}_j' = c_B^T B^{-1} A_j - c_j' =$$

$$= c_B^T B^{-1} A_j - (c_j + \Delta) = \underbrace{c_B^T B^{-1} A_j - c_j}_{\hat{c}_j} - \Delta = \hat{c}_j - \Delta$$

$\hat{c}_j \leftarrow$ reduced cost before change

The current basis is still optimal if $\hat{c}_j' \le 0 (\ge 0)$, hence

↓

Solve for Δ

$\hat{c}_j - \Delta \le 0$ $(\ge)$ Δ_{min} / Δ_{max}

where $b \ge 0$, and optimal bfs $[x^* = (x_N | x_B)]$ and the optimal tableau:

basis	x_N ($\hat{c}_N^T$)	x_B	rhs
z^*	$c_B^T B^{-1} N - c_N^T$	0^T	$c_B^T B^{-1} b$
x_B	$B^{-1} N$	I	$B^{-1} b$

OR

basis	x_{N_0}	x_{B_0}	rhs
z^*	$c_B^T B^{-1} N_0 - c_{N_0}^T$	$c_B^T B^{-1} - c_{B_0}^T$	$c_B^T B^{-1} b$
x_B	$B^{-1} N_0$	B^{-1}	$B^{-1} b$

- From the tableau:

Change in the objective function coefficient

Non-basic variable – an example

- $\min z = -x_1 - 2x_2$

s.t. $-2x_1 + x_2 + x_3 = 2$

$-x_1 + 2x_2 + \quad + x_4 = 7$

$x_1 + 1.5x_2 \quad + x_5 = 3$

$x_1,\ x_2,\ x_3,\ x_4, x_5 \geq 0$

optimal

Final tableau:

$$x_{B_0} = c_B^T B^{-1}$$

$$c_B^T B^{-1} - (c_{B_0})^T = 0$$

	B	B	B	N	N	
basis	x_1	x_2	x_3	x_4	x_5	rhs
z	0	0	0	-1	-2	-13
x_2	0	1	0	$\frac{1}{2}$	$\frac{1}{2}$	5
x_1	1	0	0	0	1	3
x_3	0	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	3

- From the final tableau: $\hat{c}_N^T = \langle -1, -2 \rangle$

$$\begin{array}{cccccc} & x_1 & x_2 & x_3 & x_4 & x_5 \\ c^T = \langle & -1 & -2 & 0 & 0 & 0 \ \rangle \end{array}$$

$x_{B_0} = (x_3\ x_4\ x_5)$; $c_{B_0}^T = (0, 0, 0)$

$$B^{-1}N = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix} \qquad B^{-1}b = \begin{pmatrix} 5 \\ 3 \\ 3 \end{pmatrix} \qquad c_B^T B^{-1} = \langle 0, -1, -2 \rangle$$

(in c_B^T the order is important)

$$B^{-1} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \\ 1 & -\frac{1}{2} & \frac{3}{2} \end{pmatrix}$$

but $b = \begin{pmatrix} 2 \\ 7 \\ 3 \end{pmatrix}$

Change in the objective function coefficient

Non-basic variable – an example

x_4 is non-basic

$c_4 = 0$; assume $c_4' = \Delta + 0.$

Final tableau:

basis	x_1	x_2	x_3	x_4	x_5	rhs
z	0	0	0	-1	-2	-13
x_2	0	1	0	$\frac{1}{2}$	$\frac{1}{2}$	5
x_1	1	0	0	0	1	3
x_3	0	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	3

Perform the sensitivity analysis:

$$\hat{c}_4' = c_B^T B^{-1} A_4 - c_4' =$$

$$= \underbrace{c_B^T B^{-1} A_4 - c_4}_{\hat{c}_4} - \Delta =$$

$$= \hat{c}_4 - \Delta$$

For $\hat{c}_4' \leq 0$, $\rightarrow \hat{c}_4 - \Delta \leq 0$

$\downarrow$

$$-1 - \Delta \leq 0$$

$$\Delta \geq -1$$

Change in the objective function coefficient

Basic variable:

➢ Suppose that $c'_B = \underbrace{c_B + \Delta_{c_B}}$,

➢ How will it affect the solution?

optimality w;

$\hat{c}_B = 0$, hence

➢ From the tableau:

$\hat{c}_N^T = c_B^T B^{-1} N - c_N^T \le (or \ \ge) 0^T;$ (1) op

min max

$x_B = B^{-1} b \ge 0;$ (2)

$x_N = 0;$ (3) fe

$z^* = c_B^T B^{-1} b$ value of OF (4)

$\hat{c}'_N = c_B'^T B^{-1} N - c_N^T =$, hence for non-basic components

$$\hat{c}'_j = \underbrace{c_B^T B^{-1} N - c_N^T}_{\hat{c}_N^T} + \Delta_{c_B} B^{-1} N =$$

$$= \hat{c}_N^T + \Delta_{c_B} B^{-1} N \quad \le 0 \ (\min) \quad \ge 0 \ (\max)$$

The current basis is still optimal if $\hat{c}'_j \le 0 (\ge 0)$, hence

Solve system of Inequalities for Δ

$$\hat{c}'_j = \hat{c}_j + \Delta_{c_B} B^{-1} A_j \quad \text{for each } x_j \in x_N$$

$$\hat{c}' = c_B'^T B^{-1} A - c^{T\prime} =$$

$$= \left(c_B'^T \underbrace{B^{-1} B}_{I} \;\middle|\; c_B'^T B^{-1} N \right) - \left(c_B'^T \;\middle|\; c_N^T \right) =$$

$$= \left(c_B'^T - c_B'^T \;\middle|\; c_B'^T B^{-1} N - c_N^T \right) =$$

$$= \left(0^T \;\middle|\; \hat{c}_N'^T \right)$$

$$\quad x_B \quad | \quad x_N$$

x_3 is basic $\quad c = \begin{pmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{matrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \end{matrix}$

Change in the objective function coefficient

Basic variable – an example

$c_3 = 0$; assume $c'_3 = \Delta + 0$.

Hence $x_B = \begin{pmatrix} x_2 \\ x_1 \\ x_3 \end{pmatrix} \to c_B = \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}$

$\boldsymbol{c}_B'^T = \boldsymbol{c}_B^T + \Delta_{c_B}$ and $\Delta_{c_B} = (\mathbf{0}, \mathbf{0}, \Delta)^T$

Final tableau:

basis	x_1	x_2	x_3	x_4	x_5	rhs
z	0	0	0	-1	-2	-13
x_2	0	1	0	$\frac{1}{2}$	$\frac{1}{2}$	5
x_1	1	0	0	0	1	3
x_3	0	0	1	-$\frac{1}{2}$	$\frac{3}{2}$	3

$\widehat{c_N'^T} = c_B'^T B^{-1} N - c_N^T =$

$= \hat{c}_N^T + \Delta_{c_B}^T B^{-1} N \leq 0$

To keep the basis optimal

$z^{*'} = c_B'^T B^{-1} b =$

$= c_B^T B^{-1} b + \Delta_{c_B}^T B^{-1} b =$

$= z^* + \Delta_{c_B}^T \underbrace{B^{-1} b}$ $=$

$= -13 + (0, 0, \Delta) \begin{pmatrix} 5 \\ 3 \\ 3 \end{pmatrix} =$

$= -13 + 3\Delta$

Hence

$(-1 \ -2) + (0, 0, \Delta) \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix} \leq 0$

$(-1, -2) + (-\tfrac{1}{2}\Delta, \tfrac{3}{2}\Delta) \leq 0$

$\begin{cases} -1 - \frac{1}{2}\Delta \leq 0 \\ -2 + \frac{3}{2}\Delta \leq 0. \end{cases} \to \begin{matrix} \Delta \geq -2 \\ \Delta \leq \frac{4}{3} \end{matrix} \to$ if $-2 \leq \Delta \leq 4/3$

bfs is still optimal

Change in the right-hand side

$\hat{c}_N$ - optimality? No

Change in rhs affects: 1) Feasibility $x_B = B^{-1}\underline{b} \rightarrow$ yes

2) OF value : $z = c_B^T B^{-1} \underline{\underline{b}}$ yes

- Let $\boldsymbol{b}' = \boldsymbol{b} + \Delta_{\boldsymbol{b}}$

- How will it affect the solution? $\boldsymbol{x}_{\boldsymbol{B}}' \geq 0$, hence

$$x_B' = B^{-1}b' = B^{-1}b + B^{-1}\Delta_b = x_B + B^{-1}\Delta_b \geq 0$$

- How will it affect the optimal objective function value?

$$z^{*\prime} = c_B^T B^{-1} \underbrace{b'}_{b+\Delta_b} = \underbrace{c_B^T B^{-1} b}_{z^*} + c_B^T B^{-1}\Delta_b$$

$$z^* + c_B^T B^{-1}\Delta_b$$

Change in the right-hand side

Example - let's change ~~the~~ RHS of second constraint by Δ:

$b' = b + \Delta_b = b + (\mathbf{0}, \Delta, \mathbf{0})^T$

$= \begin{pmatrix} 2 \\ 7 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 \\ \Delta \\ 0 \end{pmatrix}$

To satisfy $x'_B \geq 0$,

$x'_B = B^{-1} b' =$

Final tableau:

x_{B_0}

basis	x_1	x_2	x_3	x_4	x_5	rhs
z	0	0	0	-1	-2	-13
x_2	0	1	0	$\frac{1}{2}$	$\frac{1}{2}$	5
x_1	1	0	0	0	1	3
x_3	0	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	3

B^{-1}

$= B^{-1} b + B^{-1} \Delta_b =$

$= \begin{pmatrix} 5 \\ 3 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 & 1/2 & 1/2 \\ 0 & 0 & 1 \\ 1 & -1/2 & 3/2 \end{pmatrix} \begin{pmatrix} 0 \\ \Delta \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 3 \end{pmatrix} + \begin{pmatrix} 1/2\,\Delta \\ 0 \\ -1/2\,\Delta \end{pmatrix} = \begin{array}{c} x'_2 \\ x'_1 \\ x'_3 \end{array} \begin{pmatrix} 5 + 1/2\,\Delta \\ 3 \\ 3 - 1/2\,\Delta \end{pmatrix}$

$$x_B' \geq 0 \rightarrow \begin{cases} x_2' \rightarrow 5 + \frac{1}{2}\Delta \geq 0 \\ x_3' \rightarrow 3 - \frac{1}{2}\Delta \geq 0 \end{cases} \rightarrow \begin{cases} \Delta \geq -10 \\ \Delta \leq 6 \end{cases} \rightarrow$$

$x_1' = 3$

if $-10 \leq \Delta \leq 6$,

The current Basis is optimal and feas.

$$x_2' = 5 + \frac{1}{2}\Delta$$

$$x_1' = 3$$

$$x_3' = 3 - \frac{1}{2}\Delta$$

$$z^{*\prime} = z^* + C_B^T B^{-1} \Delta_b =$$

$$= -13 + \langle 0, -1, -2 \rangle \begin{pmatrix} 0 \\ \Delta \\ 0 \end{pmatrix} =$$

$$= -13 - \Delta$$

Change in a column of a constraint matrix

- Let $A_j' = A_j + \Delta_{A_j}$
- If the changed column corresponds to a non-basic variable, then the change will affect $\hat{c}_j$ only!

$x_B = B^{-1} b$ - Feas → NO

$$\hat{c}_j' = c_B^T B^{-1} A_j' - c_j = \underbrace{c_B^T B^{-1} A_j - c_j} + c_B^T B^{-1} \Delta_{A_j}$$

Hence

$$= \hat{c}_j + c_B^T B^{-1} \Delta_{A_j}$$

↓

solve inequality to keep $\hat{c}_j$ satisfying opt. condition

➢ $\min z = -x_1 - 2x_2$

s.t. $-2x_1 + x_2 + x_3 = 2$

$-x_1 + 2x_2 + \quad + x_4 = 7$

$x_1 + 1.5x_2 \quad + x_5 = 3$

$x_1,\ x_2,\ x_3,\ x_4, x_5 \geq 0$

Change in a column of a constrain

Example - let's change a_{35} (coefficient for x_5) b

$A_5' = A_5 + \Delta_{A_j} = A_5 + (0, 0, \Delta)^T$

row 3, column 5 → A_5

To satisfy $\hat{c}_j' \leq 0$,

$A_5 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$; $A_5' = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \Delta \end{pmatrix}$

Fina

basis	x_1	x_2	x_3	x_4	x_5	rhs
z	0	0	0	-1	-2	-13
x_2	0	1	0	$\frac{1}{2}$	$\frac{1}{2}$	5
x_1	1	0	0	0	1	3
x_3	0	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	3

$\hat{c}_5' = \hat{c}_5 + \underbrace{c_B^T B^{-1}} \Delta_{A_5} =$

$= -2 + (0, -1, -2)\begin{pmatrix} 0 \\ 0 \\ \Delta \end{pmatrix} = -2 - 2\Delta \leq 0$

if $\Delta \geq -1$

current Bfs is still optimal

Change in a column of a constraint matrix

- If the changed column corresponds to a basic variable, then we will have to re-calculate B^{-1}. Let $B' = B + \Delta e_j e_k^T$
- Sherman-Morrison formula:

  Suppose $A \in R^{n \times n}$ is an invertible matrix, and $u, v \in R^n$ are column vectors. Then

$$(A + uv^T)^{-1} = A^{-1} - \frac{A^{-1}uv^TA^{-1}}{1+v^TA^{-1}u} \quad \text{if and only if} \quad 1 + v^TA^{-1}u \neq 0$$

- $(B')^{-1} = B^{-1} - \frac{\Delta}{1+\Delta \mathrm{B}_{kj}^{-1}} B_{:j}^{-1} B_{k:}^{-1}$
  - Feasibility: $(B')^{-1}b \geq 0$
  - Optimality: $c_B^T(B')^{-1}N - c_N \leq 0$

Addition of a new constraint

➢ **Example:** $max\ z = 5x_1 + 4x_2$

s.t. $x_1 + x_2 \leq 10$

$x_1 \leq 4$

$x_1, x_2 \geq 0$

$x_{B_0} = (s_1, s_2)$

$c_{B_0}^T = (0, 0)$

$c_B^T B^{-1} =$

➢ Optimal tableau:

basis	x_1	x_2	s_1	s_2	rhs
z	0	0	**4**	**1**	**44**
x_2	0	1	1	-1	6
x_1	1	0	0	1	4

$\underbrace{s_1 \quad s_2}_{x_{B_0}}$

$\boldsymbol{x_B} = \langle x_2\ x_1 \rangle$

$c_B^T B^{-1} = \langle 4, 1 \rangle$

$\hat{c}_{x_{B_0}} = c_B^T B^{-1} \underset{\overset{\shortparallel}{I}}{B_0} - \underset{\overset{\shortparallel}{0}}{c_{B_0}} = c_B^T B^{-1}$

$B^{-1} =$

Addition of a new constraint

if BFS satisfies new constraint, then no change req.

- Now add another constraint:

$$\begin{aligned} & max\ z = 5x_1 + 4x_2 \\ \textbf{\textit{s.t.}}\quad & x_1 + x_2 \le 10 \\ & x_1 \le 4 \\ & x_1 + 3x_2 \le 15 \quad (*) \\ & x_1, x_2 \ge 0 \end{aligned}$$

new constraint →

- The current optimal $x_B = \begin{array}{c} x_1 \\ x_2 \end{array}\begin{pmatrix} 4 \\ 6 \end{pmatrix}$ does/does not satisfy the new constraint

- New optimal value $z^{*\prime} \le z^*$ current opt. as new feas. region is not "greater" than before.

- Hence we need to find new optimal solution:
  - $(*)$: $x_1 + 3x_2 + s_3 = 15$ s_3 is another basic variable
  - To add s_3 to basis , need to present (*) in a canonical form (that is need to eliminate x_1 and x_2)

$(x_B') = \begin{pmatrix} x_1 \\ x_2 \\ s_3 \end{pmatrix} \rightarrow$

Addition of a new constraint

- Hence we need to find new optimal solution:
  - $(*)$: $x_1 + 3x_2 + s_3 = 15$ s_3 is another
  - To add s_3 to basis , need to present (*) in a canonical form (that is need to eliminate x_1 and x_2) :

From tableau $x_2 = 6 - s_1 + s_2$

basis	x_1	x_2	s_1	s_2	rhs
z	0	0	**4**	**1**	**44**
x_2	0	1	1	-1	6
x_1	1	0	0	1	4

$$\left.\begin{aligned} x_2 + s_1 - s_2 &= 6 \\ x_1 + s_2 &= 4 \end{aligned}\right\} \rightarrow (*)$$

$$\underbrace{4 - s_2}_{x_1} + \underbrace{18 - 3s_1 + 3s_2}_{3x_2} + s_3 = 15$$

$$s_3 - 3s_1 + 2s_2 = -7$$

Addition of a new constraint

- New tableau:

	x_1	x_2	s_1	s_2	s_3	RHS
z	0	0	4	1	0	44
x_2	0	1	1	-1	0	6
x_1	1	0	0	1	0	4
s_3	0	0	$\boxed{-3}$	2	1	-7
z	0	0	0	$1+8/3$	$4/3$	$44-\frac{28}{3}$
x_2	0	1	0	$-1/3$	$1/3$	$11/3$
x_1	1	0	0	1	0	4
s_1	0	0	1	$-2/3$	$-1/3$	$7/3$

$-7 < 0 \longrightarrow s_3$ leaving

$R_0' = R_0 - 4R_3'$

$R_1' = R_1 - R_3'$

$R_2' = R_2$

$R_3' = R_3 / (-3)$

opt. as $\hat{c}_N > 0$; $z^* = \frac{104}{3} < 44$ $\quad x^* = \begin{pmatrix} 4 \\ 11/3 \end{pmatrix} \begin{matrix} x_1 \\ x_2 \end{matrix}$

Addition of a new variable

- Assume we add a new variable x_j with objective function coefficient c_j and a constraint column A_j

1. Calculate $\hat{c}_j = \underbrace{c_B^T B^{-1}}_{\text{From Tableau}} \underbrace{A_j}_{\text{given}} - \underbrace{c_j}_{\text{given}}$

- If $\underbrace{\hat{c}_j \le 0 \ (or \ \le 0)}_{\text{opt. cond. satisfied}}$, put x_j in x_N , and x_B does not change
- Otherwise x_j enters basis and the optimal *bfs* will change

Addition of a new variable

basis	x_1	x_2	s_1	s_2	rhs
z	0	0	4	1	44
x_2	0	1	1	-1	6
x_1	1	0	0	1	4

- **Example:** let's add another variable:

$$\max z = 5x_1 + 4x_2 + 8x_3$$

$$\text{s.t.} \quad x_1 + x_2 + x_3 \leq 10$$

$$x_1 + 2x_3 \leq 4$$

$$x_1, x_2, x_3 \geq 0$$

x_3 is new variable

- $c_3 = 8$ $\quad A_3 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\hat{c}_3 = c_B^T B^{-1} A_3 - c_3 =$

$= \langle 4, 1 \rangle \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 8 = 6 - 8 = -2 < 0$

- The current $x_B = (x_2 \; x_1)$ is/is not optimal and x_3

- Calculating $\hat{A}_3 = B^{-1} A_3 =$

$$= \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

curren Bfs not opt

↓

x_3 enters basis

Addition of a new variable

- New tableau:

basis	x_1	x_2	s_1	s_2	x_3	RHS
z	0	0	4	1	$\boxed{-2}$	44
x_2	0	1	1	-1	-1	6
x_1	1	0	0	1	$\boxed{2}$	4

z	1	0	4	2	0	48
x_2	1/2	1	1	-1/2	0	8
x_3	1/2	0	0	1/2	1	2

$R_0' = R_0 + R_2'$

$R_1' = R_1 + R_2'$

$R_2' = \frac{R_2}{2}$

optimal; $z^* = 48 > 44$

$X^* = \begin{matrix} x_2 \\ x_3 \end{matrix} \begin{pmatrix} 8 \\ 2 \end{pmatrix}$

$c = \begin{pmatrix} 10 \\ 7 \\ 6 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

Big Example

➢ Consider

$$max\ z = 10x_1 + 7x_2 + 6x_3$$

$$s.t. \quad 3x_1 + 3x_2 + x_3 \le 36$$
$$x_1 + x_2 + 2x_3 \le 32$$
$$2x_1 + x_2 + x_3 \le 22$$
$$x_1, x_2, x_3 \ge 0$$

$X_{B_0} = (s_1\ \ s_2\ \ s_3)$

$X_B = (x_2\ \ s_2\ \ x_3)$

$C_{B_0} = (0,\ 0,\ 0)$

Optimal tableau:

	N	B	B	N	B	N	
				B_0	B_0	B_0	
basis	x_1	x_2	x_3	s_1	s_2	s_3	rhs
z	3	0	0	1	0	5	146
x_2	1	1	0	1	0	-1	14
s_2	-2	0	0	1	1	-3	2
x_3	1	0	1	-1	0	2	8

$\hat{c}_N^T = \begin{array}{c} \begin{matrix} x_1 & s_1 & s_3 \end{matrix} \\ (3,\ 1,\ 5) \end{array}$

$\underline{c_B^T B^{-1}} = (1\ \ 0\ \ 5)$

$\underbrace{B^{-1}N} = \begin{pmatrix} 1 & 1 & -1 \\ -2 & 1 & -3 \\ 1 & -1 & 2 \end{pmatrix}$

$\hat{b} = B^{-1}b = \begin{pmatrix} 14 \\ 2 \\ 8 \end{pmatrix}$

$B^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & -3 \\ -1 & 0 & 2 \end{pmatrix}$

Big Example

By how much can the objective coefficient of x_1 be changed without altering the optimal basis?

basis	x_1	x_2	x_3	s_1	s_2	s_3	rhs
z	3	0	0	1	0	5	146
x_2	1	1	0	1	0	-1	14
s_2	-2	0	0	1	1	-3	2
x_3	1	0	1	-1	0	2	8

1. x_1 is non-basic

$\hat{c}_1' = \hat{c}_1 - \Delta = 3 - \Delta \geq 0 \rightarrow$ so x_B is still optimal

$\downarrow$

$\Delta \leq 3$

Big Example

- By how much can the objective coefficient of x_2 be changed without altering the optimal basis?

basis	x_1	x_2	x_3	s_1	s_2	s_3	rhs
z	3	0	0	1	0	5	146
x_2	1	1	0	1	0	-1	14
s_2	-2	0	0	1	1	-3	2
x_3	1	0	1	-1	0	2	8

$x_B = (x_2 \; s_2 \; x_3)$

$c_2' = c_2 + \Delta$

x_2 is BASIC → will affect all $\hat{c}_N$

$$c_B' = c_B + \Delta_{c_B} = \begin{array}{c} c_2 \\ c_{s_2} \\ c_3 \end{array} \begin{pmatrix} 7 \\ 0 \\ 6 \end{pmatrix} + \begin{pmatrix} \Delta \\ 0 \\ 0 \end{pmatrix}$$

$$\hat{c}_N^{T\,\prime} = \hat{c}_N^T + \Delta_{c_B}^T B^{-1} N = (3, 1, 5) + (\Delta \;\; 0 \;\; 0) \begin{pmatrix} 1 & 1 & -1 \\ -2 & 1 & -3 \\ 1 & -1 & 2 \end{pmatrix}$$

$$= (3, 1, 5) + (\Delta, \Delta, -\Delta) = (\underset{\substack{\vee\!/ \\ 0}}{3+\Delta}, \underset{\substack{\vee\!/ \\ 0}}{1+\Delta}, \underset{\substack{\vee\!/ \\ 0}}{5-\Delta})$$

$$\begin{cases} 3+\Delta \geq 0 \\ 1+\Delta \geq 0 \\ 5-\Delta \geq 0 \end{cases} \rightarrow \begin{cases} \Delta \geq -3 \\ \Delta \geq -1 \\ \Delta \leq 5 \end{cases} \rightarrow \text{if } -1 \leq \Delta \leq 5$$

current Bfs is still optimal

$$\hat{c}_N'^T = c_B'^T B^{-1} N - c_N^T = (c_B^T + \Delta^T) B^{-1} N - c_N^T =$$

$$= \underbrace{c_B^T B^{-1} N - c_N^T} + \Delta^T B^{-1} N =$$

$$= \hat{c}_N^T + \Delta^T B^{-1} N \geq 0$$

opt. condition for max

Big Example

- By how much can the *rhs* of the first constraint be changed without altering the optimal basis?

basis	x_1	x_2	x_3	s_1	s_2	s_3	rhs
z	3	0	0	1	0	5	146
x_2	1	1	0	1	0	-1	14
s_2	-2	0	0	1	1	-3	2
x_3	1	0	1	-1	0	2	8

$$b' = b + \begin{pmatrix} \Delta \\ 0 \\ 0 \end{pmatrix}$$

$$x_B' = x_B + B^{-1}\begin{pmatrix} \Delta \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 14 \\ 2 \\ 8 \end{pmatrix} + \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & -3 \\ -1 & 0 & 2 \end{pmatrix}\begin{pmatrix} \Delta \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 14+\Delta \\ 2+\Delta \\ 8-\Delta \end{pmatrix} \geq 0 \rightarrow$$

$$x_2 = 14 + \Delta \geq 0$$
$$s_2 = 2 + \Delta \geq 0 \rightarrow$$
$$x_3 = 8 - \Delta \geq 0$$

$$-2 \leq \Delta \leq 8$$

$$z^{*\prime} = z^* + c_B^T B^{-1} \Delta_b = 146 + (1\ 0\ 5)\begin{pmatrix} \Delta \\ 0 \\ 0 \end{pmatrix} = 146 + \Delta$$

Big Example

- By how much can the coefficient of x_1 in the first constraint be changed without altering the optimal basis?

basis	x_1	x_2	x_3	s_1	s_2	s_3	rhs
z	3	0	0	1	0	5	146
x_2	1	1	0	1	0	-1	14
s_2	-2	0	0	1	1	-3	2
x_3	1	0	1	-1	0	2	8

x_1 is non basic $\rightarrow$ only $\hat{c}_1$ will be affected

$$A_1' = A_1 + \begin{pmatrix} \Delta \\ 0 \\ 0 \end{pmatrix}$$

$$\hat{c}_1' = \hat{c}_1 + c_B^T B^{-1} \begin{pmatrix} \Delta \\ 0 \\ 0 \end{pmatrix} = 3 + \begin{pmatrix} 1 & 0 & 5 \end{pmatrix} \begin{pmatrix} \Delta \\ 0 \\ 0 \end{pmatrix} =$$

$$= 3 + \Delta \geq 0$$

$$\downarrow$$

$$\Delta \geq -3$$

- Challenge add a constraint

$$x_2 - x_3 \leq 10$$

- add x_4 $c_4 = 2$

$$A_4 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$