Introduction to Optimisation:

Sensitivity analysis

Lecture 7

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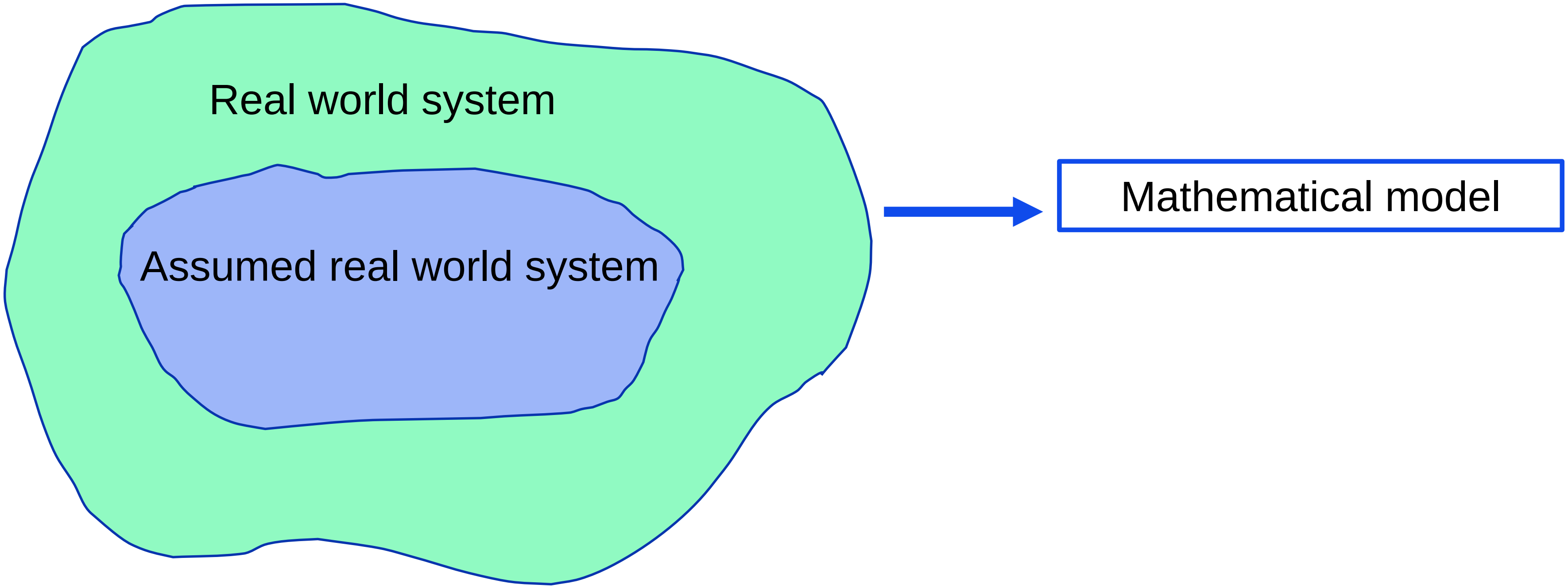
Introduction

Mathematical Optimization modelling - some assumptions:

- Linearity of constraints and functions
- Data certainty
- Values of parameters



Introduction

Sensitivity analysis is a systematic study of how sensitive the LP's optimal solution is to (small) changes in the LP's parameters and it is presented to give answers to questions of the following forms:

1. If the objective function changes, how does the optimal solution change?

2. If the amount of resources available changes, how does the optimal solution change?

3. If an additional constraint is added to the LP, how does the optimal solution change?

Sensitivity analysis may allow to avoid re-solving an LP if the change in parameters does not imply the change in optimal basis.

Introduction

- Consider $\min\ (\boldsymbol{or\ max})\ z = \boldsymbol{c}^{\boldsymbol{T}}\boldsymbol{x}$

$$s.t. \quad \boldsymbol{Ax} = \boldsymbol{b},$$
$$\boldsymbol{x} \geq \mathbf{0},$$

where $\boldsymbol{b} \geq \mathbf{0}$, and optimal bfs $|\boldsymbol{x}^{*\boldsymbol{T}} = (\boldsymbol{x}_{\boldsymbol{N}}^{\boldsymbol{T}} | \boldsymbol{x}_{\boldsymbol{B}}^{\boldsymbol{T}})$ and the optimal tableau**:**

basis	$\mathbf{x_N}$	$\mathbf{x_B}$	rhs
z^*	$\mathbf{c_B^T B^{-1} N - c_N^T}$	0^T	$\mathbf{c_B^T B^{-1} b}$
$\mathbf{x_B}$	$\mathbf{B^{-1} N}$	$\mathbf{I}$	$\mathbf{B^{-1} b}$

OR

basis	$\mathbf{x_{N_0}}$	$\mathbf{x_{B_0}}$	rhs
z^*	$\mathbf{c_B^T B^{-1} N_0 - c_{N_0}^T}$	$\mathbf{c_B^T B^{-1} - c_{B_0}^T}$	$\mathbf{c_B^T B^{-1} b}$
$\mathbf{x_B}$	$\mathbf{B^{-1} N_0}$	$\mathbf{B^{-1}}$	$\mathbf{B^{-1} b}$

- From the tableau:

$\bar{c}_N^T = c_B^T B^{-1} N - c_N^T \leq\ (or\ \geq) 0^T;$ (1) *optimality condition*

$x_B = B^{-1} b \geq 0;$ (2)

$x_N = 0;$ (3) — (2), (3): *feasibility conditions*

$z^* = c_B^T B^{-1} b$ (4)

Change in the objective function coefficient

Non-basic variable:

- Let x_j be non-basic variable in optimal *bfs*, and change $c_j, j \in N$:

$$c_j' = c_j + \Delta.$$

- How will it affect the solution?

$$\widehat{c_j'} = c_B^T B^{-1} A_j - c_j' =$$

The current basis is still optimal if $\widehat{c_j'} \le 0 (\ge 0)$, hence

Solution stability:

Change in the objective function coefficient

Non-basic variable – an example

Final tableau:

basis	x_1	x_2	x_3	x_4	x_5	rhs
z	0	0	0	-1	-2	-13
x_2	0	1	0	$\frac{1}{2}$	$\frac{1}{2}$	5
x_1	1	0	0	0	1	3
x_3	0	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	3

- $\min z = -x_1 - 2x_2$

s.t. $-2x_1 + x_2 + x_3 = 2$

$-x_1 + 2x_2 + \quad + x_4 = 7$

$x_1 + 1.5x_2 \quad + x_5 = 3$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

- From the final tableau: $c_N^T =$

$B^{-1}N =$

$B^{-1}b =$

$c_B^T B^{-1} =$

(in c_B^T the order is important)

$B^{-1} =$

Change in the objective function coefficient

Non-basic variable – an example

$c_4 = 0$; assume $c_4' = \Delta + 0$.

Perform the sensitivity analysis:

$\hat{c}_4' =$

For $\hat{c}_4' \leq 0$,

Final tableau:

basis	x_1	x_2	x_3	x_4	x_5	rhs
z	0	0	0	-1	-2	-13
x_2	0	1	0	$\frac{1}{2}$	$\frac{1}{2}$	5
x_1	1	0	0	0	1	3
x_3	0	0	1	-$\frac{1}{2}$	$\frac{3}{2}$	3

Change in the objective function coefficient

Basic variable:

- Suppose that $\boldsymbol{c}'_{\boldsymbol{B}} = \boldsymbol{c}_{\boldsymbol{B}} + \Delta_{\boldsymbol{c}_{\boldsymbol{B}}}$,
- How will it affect the solution?

$\widehat{\boldsymbol{c}_{\boldsymbol{B}}} =$, hence

$\widehat{\boldsymbol{c}'_{\boldsymbol{N}}} =$, hence for non-basic components

$\widehat{c'_j} =$

The current basis is still optimal if $\widehat{c'_j} \leq 0 (\geq 0)$, hence

Change in the objective function coefficient

Basic variable – an example

$c_3 = 0$; assume $c_3' = \Delta + 0$.

Hence

$\boldsymbol{c}_B'^T = \boldsymbol{c}_B^T + \Delta_{c_B}$ and $\Delta_{c_B} = (\mathbf{0}, \mathbf{0}, \Delta)^T$

$\widehat{\boldsymbol{c}_N'^T} =$

Hence

Final tableau:

basis	x_1	x_2	x_3	x_4	x_5	rhs
z	0	0	0	-1	-2	-13
x_2	0	1	0	$\frac{1}{2}$	$\frac{1}{2}$	5
x_1	1	0	0	0	1	3
x_3	0	0	1	-$\frac{1}{2}$	$\frac{3}{2}$	3

Change in the right-hand side

Change in rhs affects: 1)

2)

- Let $\boldsymbol{b}' = \boldsymbol{b} + \Delta_{\boldsymbol{b}}$

- How will it affect the solution? $\boldsymbol{x}'_{\boldsymbol{B}} \geq 0$, hence

- How will it affect the optimal objective function value?

Change in the right-hand side

Example - let's change the second constraint by Δ:

$$\boldsymbol{b}' = \boldsymbol{b} + \Delta_{\boldsymbol{b}} = \boldsymbol{b} + (\boldsymbol{0}, \Delta, \boldsymbol{0})^{\boldsymbol{T}}$$

Final tableau:

basis	x_1	x_2	x_3	x_4	x_5	rhs
z	0	0	0	-1	-2	-13
x_2	0	1	0	$\frac{1}{2}$	$\frac{1}{2}$	5
x_1	1	0	0	0	1	3
x_3	0	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	3

To satisfy $\boldsymbol{x}'_{\boldsymbol{B}} \geq 0$,

$$\boldsymbol{x}'_{\boldsymbol{B}} =$$

Change in a column of a constraint matrix

- Let $A_j' = A_j + \Delta_{A_j}$
- If the changed column corresponds to a non-basic variable, then the change will affect

Hence

Change in a column of a constraint matrix

Example - let's change a_{35} (coefficient for x_5) by Δ:

$\boldsymbol{A}_5' = \boldsymbol{A}_5 + \Delta_{A_j} = \boldsymbol{A}_5 + (\boldsymbol{0}, \boldsymbol{0}, \Delta)^{\boldsymbol{T}}$

Final tableau:

basis	x_1	x_2	x_3	x_4	x_5	rhs
z	0	0	0	-1	-2	-13
x_2	0	1	0	$\frac{1}{2}$	$\frac{1}{2}$	5
x_1	1	0	0	0	1	3
x_3	0	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	3

To satisfy $\widehat{c_j'} \leq 0$,

Change in a column of a constraint matrix

- If the changed column corresponds to a basic variable, then we will have to re-calculate B^{-1}. Let $B' = B + \Delta e_j e_k^T$
- Sherman-Morrison formula:

  Suppose $A \in R^{n\times n}$ is an invertible matrix, and $u, v \in R^n$ are column vectors. Then

$$(A + uv^T)^{-1} = A^{-1} - \frac{A^{-1}uv^T A^{-1}}{1+v^T A^{-1}u} \quad \text{if and only if} \quad 1 + v^T A^{-1}u \neq 0$$

- $(B')^{-1} = B^{-1} - \frac{\Delta}{1+\Delta \mathrm{B}_{kj}^{-1}} B_{:j}^{-1} B_{k:}^{-1}$
  - Feasibility: $(B')^{-1}b \geq 0$
  - Optimality: $c_B^T (B')^{-1} N - c_N \leq 0$

Addition of a new constraint

- **Example:** $max\ z = 5x_1 + 4x_2$

  s.t. $x_1 + x_2 \leq 10$

  $x_1 \leq 4$

  $x_1, x_2 \geq 0$

- Optimal tableau:

basis	x_1	x_2	s_1	s_2	rhs
z	0	0	**4**	**1**	**44**
x_2	0	1	1	-1	6
x_1	1	0	0	1	4

$\boldsymbol{x_B} =$

$c_B^T B^{-1} =$

$B^{-1} =$

Addition of a new constraint

- Now add another constraint:

$$\max z = 5x_1 + 4x_2$$

$$\begin{aligned} \textbf{\textit{s.t.}} \quad & x_1 + x_2 \le 10 \\ & x_1 \le 4 \\ & x_1 + 3x_2 \le 15 \\ & x_1, x_2 \ge 0 \end{aligned}$$

- The current optimal $x_B =$ does/does not satisfy the new constraint

- New optimal value

- Hence we need to find new optimal solution:
  - $(*)$: $x_1 + 3x_2 + s_3 = 15$ s_3 is another
  - To add s_3 to basis , need to present (*) in a canonical form (that is need to eliminate x_1 and x_2)

Addition of a new constraint

➢ Hence we need to find new optimal solution:

- $(*)$: $x_1 + 3x_2 + s_3 = 15$ s_3 is another
- To add s_3 to basis , need to present (*) in a canonical form (that is need to eliminate x_1 and x_2) :

Addition of a new constraint

- New tableau:

Addition of a new variable

- Assume we add a new variable x_j with objective function coefficient c_j and a constraint column A_j

- If $\hat{c}_j \leq 0$ $(or\ \leq 0)$, put x_j in , and does not change

- Otherwise x_j enters and the optimal *bfs* will

Addition of a new variable

- **Example:** let's add another variable:

$$\max z = 5x_1 + 4x_2 + 8x_3$$

$$\begin{aligned} s.t. \quad & x_1 + x_2 + x_3 \le 10 \\ & x_1 \quad\quad + 2x_3 \le 4 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

- $c_3 =$ $A_3 =$ and $\hat{c}_3 =$

- The current $x_B =$ is/is not optimal and x_3
- Calculating $\hat{A}_3 =$

Addition of a new variable

- New tableau:

Big Example

- Consider

$$\max z = 10x_1 + 7x_2 + 6x_3$$

$$\textbf{s.t.} \quad 3x_1 + 3x_2 + x_3 \le 36$$
$$x_1 + x_2 + 2x_3 \le 32$$
$$2x_1 + x_2 + x_3 \le 22$$
$$x_1, x_2, x_3 \ge 0$$

Optimal tableau:

basis	x_1	x_2	x_3	s_1	s_2	s_3	rhs
z	3	0	0	1	0	5	146
x_2	1	1	0	1	0	-1	14
s_2	-2	0	0	1	1	-3	2
x_3	1	0	1	-1	0	2	8

$$\widehat{c_N^T} =$$

$$c_B^T B^{-1} =$$

$$B^{-1}N =$$

$$B^{-1}b =$$

$$B^{-1} =$$

Big Example

By how much can the objective coefficient of x_1 be changed without altering the optimal basis?

basis	x_1	x_2	x_3	s_1	s_2	s_3	rhs
z	3	0	0	1	0	5	146
x_2	1	1	0	1	0	-1	14
s_2	-2	0	0	1	1	-3	2
x_3	1	0	1	-1	0	2	8

Big Example

- By how much can the objective coefficient of x_2 be changed without altering the optimal basis?

basis	x_1	x_2	x_3	s_1	s_2	s_3	rhs
z	3	0	0	1	0	5	146
x_2	1	1	0	1	0	-1	14
s_2	-2	0	0	1	1	-3	2
x_3	1	0	1	-1	0	2	8

Big Example

- By how much can the *rhs* of the first constraint be changed without altering the optimal basis?

basis	x_1	x_2	x_3	s_1	s_2	s_3	rhs
z	3	0	0	1	0	5	146
x_2	1	1	0	1	0	-1	14
s_2	-2	0	0	1	1	-3	2
x_3	1	0	1	-1	0	2	8

Big Example

- By how much can the coefficient of x_1 in the first constraint be changed without altering the optimal basis?

basis	x_1	x_2	x_3	s_1	s_2	s_3	rhs
z	3	0	0	1	0	5	146
x_2	1	1	0	1	0	-1	14
s_2	-2	0	0	1	1	-3	2
x_3	1	0	1	-1	0	2	8