Introduction to Optimisation:

Constrained Nonlinear Programming

Lecture 9

Lecture notes by Dr. Julia Memar and Dr. Hanyu Gu and with an acknowledgement to Dr.FJ Hwang and Dr.Van Ha Do

Introduction

- Let $f(x)$ is be nonlinear function of vector $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$ defined over the domain $D \subseteq R^n$. Consider an NLP problem

$$\min z = f(x)$$

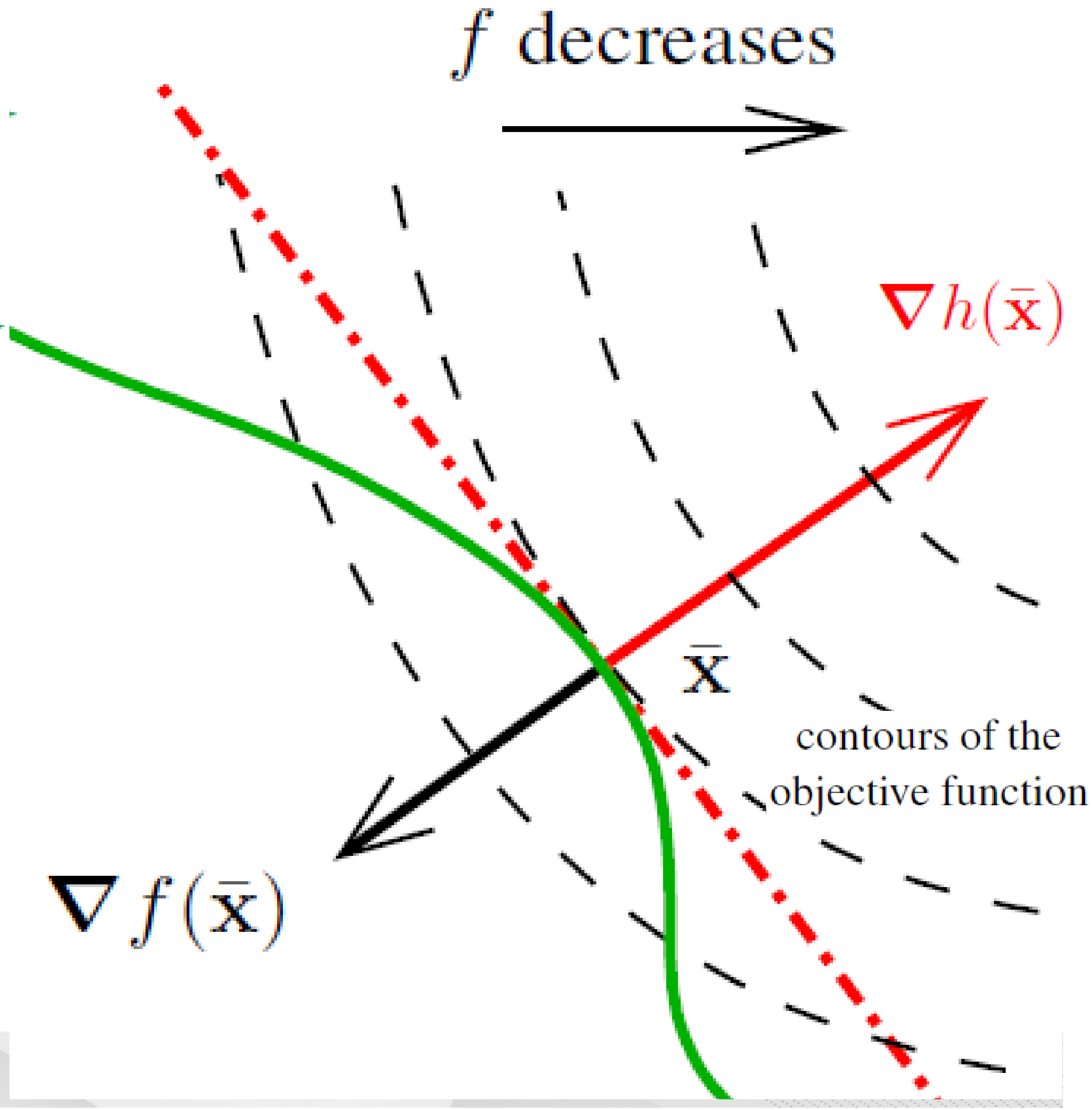
$$\text{s.t. } Ax = b,$$

where A is $m \times n$ matrix, $rank\ A = m$

- Assume that $f(x)$ has continuous second-order partial derivatives for each point $x = (x_1, x_2, \dots, x_n)$ in $D =$

Introduction

Improving direction and
Feasible direction:



Preliminaries

- Null space of $A_{m\times n}$, $n \geq m$ is

$$N(A) = \{p: Ap = 0\}$$

- Range space of $A_{m\times n}$

$$R(A) = \{q \in \Re^n: q = A^T\Lambda, \Lambda \in \Re^m\}$$

- $N(A)$ and $R(A^T)$ are orthogonal subspaces: for $q \in R(A)$ and $p \in N(A)$:

$$q^T p = \Lambda A p = 0$$

- Any $x \in \Re^n$: $x = p + q$

Reduced function

Example:

- $\min f(x_1, x_2) = x_1^2 + x_2^2$

  s.t. $3x_1 + 2x_2 = 6$

Reduced function

In general form: $Ax = b$

- Choose x_B , then $x =$

- $x =$ (particular and homogeneous solutions)

Reduced function

- Reduced cost function is

- The matrix Z is of the null space of $A_{m \times n}$

Null-space basis matrix for $A_{m \times n}$ is the $n \times (n-m)$ matrix Z:

- In other words, if $\mathrm{A}\bar{x} = b$, then any feasible point $x = \bar{x} + p$, where $p \in N(A)$
- Hence Zx_n and $-Zx_n$ are all possible feasible directions for an arbitrary x_n.

Reduced function

Example:

- $\min f(x_1, x_2, x_3) = x_1^2 + 4x_1x_3 + x_2^2$

$$s.t. \qquad 2x_1 + x_2 + 4x_3 = 5$$
$$3x_1 + x_2 - x_3 = 1$$

- The feasible set is

Reduced function

Example:

- $\min f(x_1, x_2, x_3) = x_1^2 + 4x_1x_3 + x_2^2$

$$\begin{aligned} s.t. \quad & 2x_1 + x_2 + 4x_3 = 5 \\ & 3x_1 + x_2 - x_3 = 1 \end{aligned}$$

- OR

Optimality conditions

- Unconstrained NLP problem with a reduced function:

$$\min \phi(x_N)$$

where $\phi(x_N) = f($

- To set optimality conditions find

1. *Reduced gradient* $\nabla \phi(x_N) =$
2. *Reduced Hessian* $\nabla^2 \phi(x_N) =$

Optimality conditions

Theorem 1. (Second-order necessary conditions – Linear equality constraints)

- If x^* a local minimiser of $f(x)$ over the set $\{x : Ax = b\}$, and Z is a basis matrix for the null-space of A, then

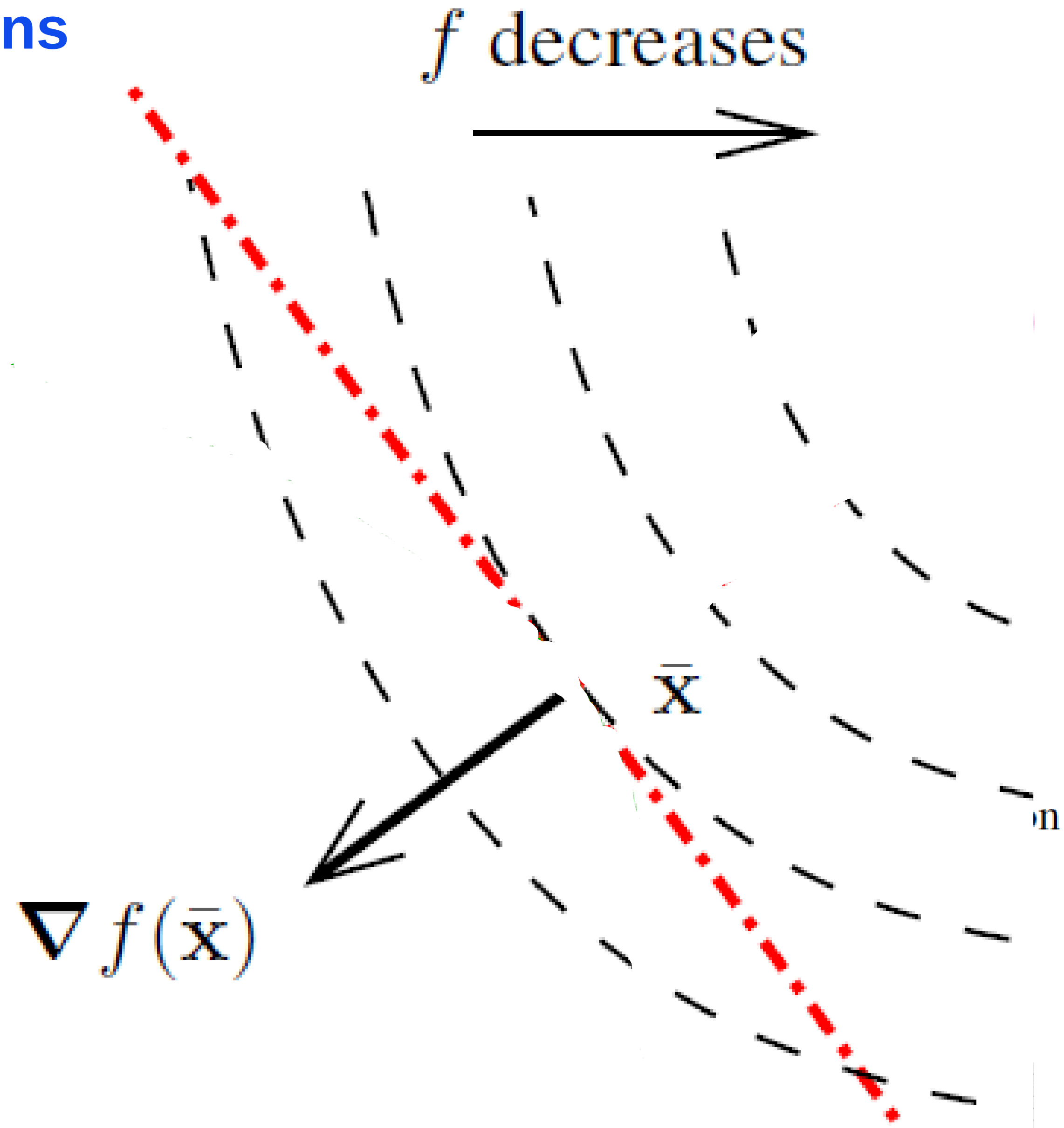
  i. $Z^T \nabla f(x^*) = 0$, and

  ii. $Z^T \nabla^2 f(x^*) Z$ is positive semidefinite.

Note: the theorem comes directly by applying Taylor's theorem on feasible directions

Optimality conditions

$\nabla f(\bar{x})$ is __________ to $N(A)$



Optimality conditions

Theorem 2. (Second-order sufficient conditions – Linear equality constraints)

- If Z is a basis matrix for the null-space of A and the point x^* satisfies

  i. $Ax^* = b$
  ii. $Z^T \nabla f(x^*) = 0$, and
  iii. $Z^T \nabla^2 f(x^*) Z$ is positive-definite.

then x^* a local minimiser of $f(x)$ over the set $\{x : Ax = b\}$.

Observe that given a point x for a considered linear-equality constrained NLP problem we can apply directly the above two theorems without deriving a reduced function.

Optimality conditions - example

- $\min f(x_1, x_2, x_3) = x_1^2 - 2x_1 + x_2^2 - x_3^2 + 4x_3$

  s.t. $\quad x_1 - x_2 + 2x_3 = 2$

- $\nabla f(x) = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$ $\quad \nabla^2 f(x) = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$

- $x_N =$ $\quad x_B =$ $\quad N =$ $\quad B =$

- $Z = \begin{pmatrix} B^{-1}N \\ I \end{pmatrix} \quad =$

- $Z^T \nabla f(x) = \qquad = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Optimality conditions - example

- Solve the resultant system of equations:

- $x^* =$ Check second order condition (iii) for x^*:

$Z^T \nabla^2 f(x^*) Z =$

with eigenvalues:

Lagrangian function – preliminaries

- Let x^* a local minimiser of $f(x)$ over the set $\{x : Ax = b\}$, and Z is a basis matrix for the null-space of A. Then $\nabla f(x^*) = Zv^* + A^T\Lambda^*$. Hence

$$\nabla f(x^*) =$$

where $\Lambda = (\lambda_1, \dots, \lambda_m)$ is a vector of Lagrangian multipliers

Lagrangian function – equality constraints

- Consider an NLP problem

$$\min z = f(x)$$

$$s.t. \quad g_i(x) = b_i, i = 1..m \qquad (**)$$

- Introduce the Lagrangian function with Lagrangian multipliers $\Lambda = (\lambda_1, \dots, \lambda_m)$

$L(x, \Lambda) =$

Lagrangian function – equality constraints

- Assume that (x^*, Λ^*) minimazes $L(x, \Lambda)$. Then at (x^*, Λ^*)

$$\frac{\partial L(x, \Lambda)}{\partial \lambda_i} = \qquad\qquad\qquad = 0, i = 1..m$$

Hence x^* does/does not satisfy (**).

To show that x^* is optimal , consider any feasible x':

Summary: If (x^*, Λ^*) minimazes $L(x, \Lambda)$, then x^* is ______________________________

Example 1

Consider the NLP problem:

- $\min f(x_1, x_2) = x_1^2 + 2x_2^2$

$$s.t. \quad x_1^2 + x_2^2 = 1$$

a) Write the Lagrangian function for this problem.

b) Use the Lagrangian to find local minimiser(s) for the given problem

Example 1

Consider the NLP problem:

- $\min f(x_1, x_2) = x_1^2 + 2x_2^2$

  $$s.t. \quad x_1^2 + x_2^2 = 1$$

- $L(x_1, x_2, \lambda) =$

- $\nabla L(x_1, x_2, \lambda) = \qquad \Rightarrow \left\{ \begin{array}{l} \\ \\ \\ \end{array} \right.$

Lagrangian function – equality constraints

- The first-order optimality condition for unconstrained NLP requires that

$\nabla L(x, \Lambda) =$ $\quad$ $i.e. \nabla_{\Lambda} L(x, \Lambda) =$ $\quad$ and $\quad$ $\nabla_x L(x, \Lambda) =$ $\quad$ (***)

$$\nabla_x L(x, \Lambda) = \qquad \Leftrightarrow \nabla f(x) =$$

- Any point (x', Λ') satisfying (***) is a s................ point for $L(x, \Lambda)$ and a feasible point for (**).

Lagrangian function – equality constraints

Theorem 3.

- If (x^*, Λ^*) is a stationary point to $L(x, \Lambda)$:

1. $\frac{\partial L(x,\Lambda)}{\partial \lambda_i} = 0, i = 1..m$

2. $\frac{\partial L(x,\Lambda)}{\partial x_j} = 0, j = 1..n$

3. *E*ach $g_i(x)$ is linear *<u>And</u>* *f(x)* is a convex function,

then x^* is a local minimum of *f(x) on* $\{g(x) = b\}$

Example 2

Consider the NLP problem:

- $\min f(x_1, x_2, x_3) = x_1^2 - 2x_1 + x_2^2 - x_3^2 + 4x_3$

s.t. $x_1 - x_2 + 2x_3 = 2$

a) Write the Lagrangian function for this problem.

b) Use the Lagrangian to find local minimiser(s) for the given problem

Example 2

Consider the NLP problem:

- $\min f(x_1, x_2, x_3) = x_1^2 - 2x_1 + x_2^2 - x_3^2 + 4x_3$

s.t. $\quad x_1 - x_2 + 2x_3 = 2$

- $\nabla L(x_1, x_2, x_3, \lambda) = \quad \Longrightarrow \quad \left\{ \right.$

Example 3*

Consider the NLP problem:

- $\min f(x_1, x_2, x_3) = 3x_1^2 - \frac{1}{2}x_2^2 - \frac{1}{2}x_3^2 + x_1x_2 - x_1x_3 + 2x_2x_3$

s.t. $2x_1 - x_2 + x_3 = 2$

a) Write the Lagrangian function for this problem.

b) Use the Lagrangian to find local minimiser(s) for the given problem

* from Linear and Non-Linear Programming by S.G.Nash and A.Sofer

Example 3

Consider the NLP problem:

- $\min f(x_1, x_2, x_3) = 3x_1^2 - \frac{1}{2}x_2^2 - \frac{1}{2}x_3^2 + x_1x_2 - x_1x_3 + 2x_2x_3$

s.t. $2x_1 - x_2 + x_3 = 2$

- $\nabla L(x_1, x_2, x_3, \lambda) = \quad \Longrightarrow \quad \left\{ \right.$