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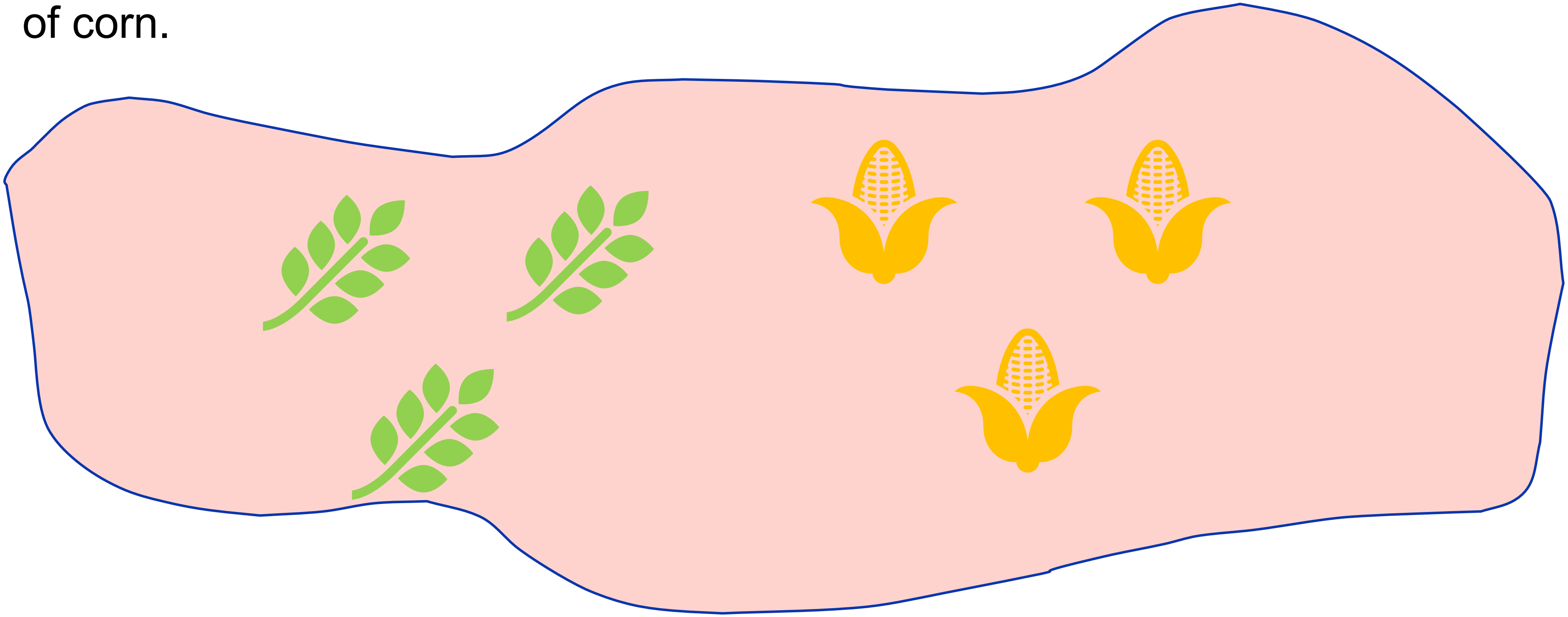
Introduction to Optimisation

Lecture 1

Lecture notes by Dr. Julia Memar and Dr. Hanyu Gu and with an acknowledgement to Dr.FJ Hwang and Dr.Van Ha Do

Let's start with an example…

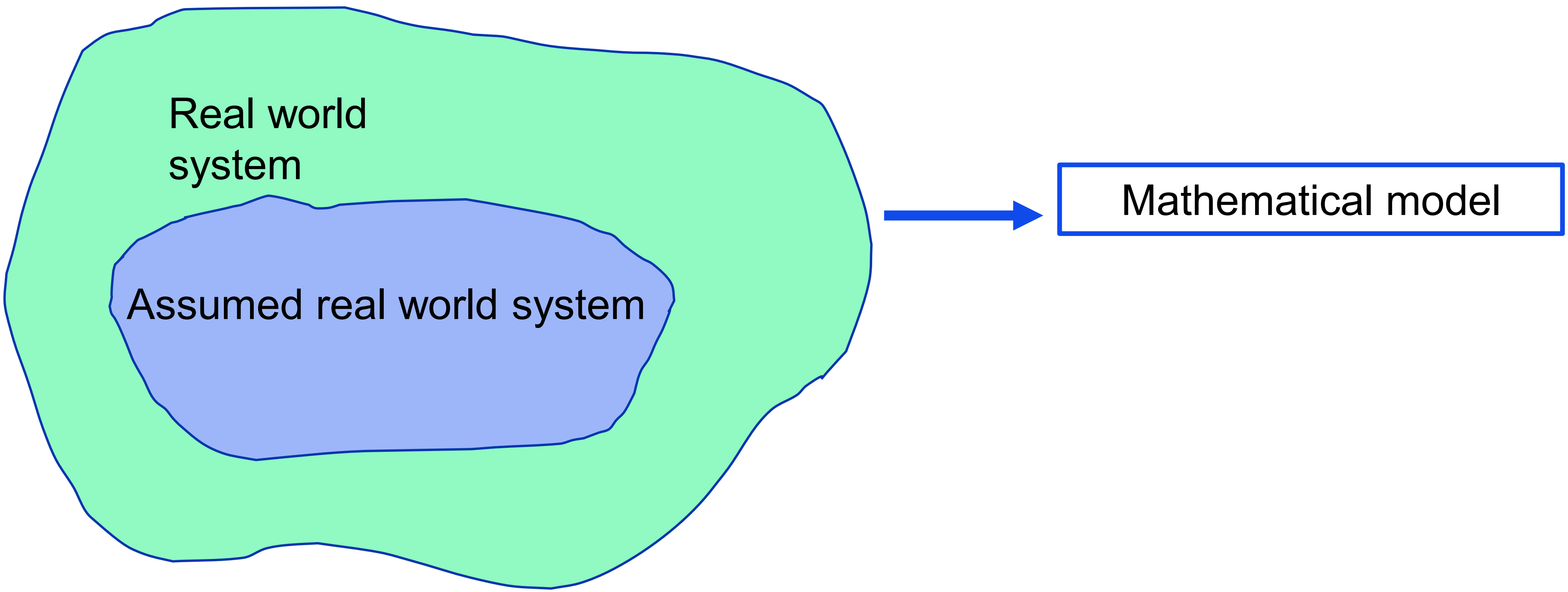
Farmer Jones plants wheat and corn 4 acres of land. The selling prices are $8 and $5 per tonne of wheat and corn, respectively. The yields per acre are 2 tonnes and 3 tonnes for wheat and corn, respectively. The farm cannot sell more than 4 tonnes of wheat and more than 9 tonnes of corn.



What is the maximal revenue that the farm can achieve?

What is Mathematical Optimisation?

- Mathematical optimisation (or mathematical programming) is a field of applied mathematics that is concerned with solution of quantitative problems.



What is Mathematical Optimisation?

- The general form of the mathematical programming problems is the following:

OF

$$\max(or\ min) \qquad z = f(x)$$

objective function

s.t.

$$subject\ to\ \ g(x) \leq (or\ \geq, or\ =)\ 0,$$

constraints

DV

decision variables

$$x \geq 0, (or\ \leq 0, or\ urs - unrestricted\ in\ sign)$$

where x and $g(x)$ are vectors.

What is Mathematical Optimisation?

Example:

$$\max z = f(x_1, x_2, x_3, x_4) \quad \text{Objective function}$$

s.t.

$$\left.\begin{aligned} g_1(x_1, x_2, x_3, x_4) &\le 0, \\ g_2(x_1, x_2, x_3, x_4) &\le 0, \\ g_3(x_1, x_2, x_3, x_4) &\le 0, \\ g_4(x_1, x_2, x_3, x_4) &\le 0, \end{aligned}\right\} \text{Constraints}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

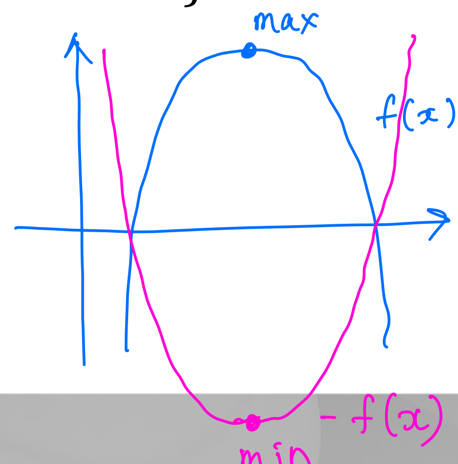
$\downarrow$ vector of DV

$x \in R^4$

The set $S = \{x : g(x) \le 0 \; and \; x \ge 0\}$ is called ***the feasible region.***

Observe that

$$z = \max_{x \in S} f(x) = -\min_{x \in S}(-f(x))$$



What is Mathematical Optimisation?

Examples:

OF → $max\ 2x_1 + 3x_2$ ← linear $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

s.t. constr. $\begin{cases} x_1 + 3x_2 \leq 6 \\ 3x_1 + 5x_2 \leq 15 \end{cases}$ LP

D.V. $\{ x_1, x_2 \geq 0$

Linear program

not linear

OF → $min\ x_1^3 + 2x_1^2 x_2 + 5x_2^4$

NON-Linear program

s.t. const. $\begin{cases} 2x_1 + x_2 \geq 4 \\ 3x_1^2 + 2x_2 \geq 5 \end{cases}$

NLP

D.V. $\{ x_1, x_2 \geq 0$

OF → $max\ 2x_1 - 2x_1x_2 + 5x_2$

s.t. const. $x_1 + x_2 \leq 4$

Integer programming

DV $x_1, x_2 \geq 0,\ x_1, x_2\ integer$

IP

How to solve an optimization problem?

Formulate the problem:

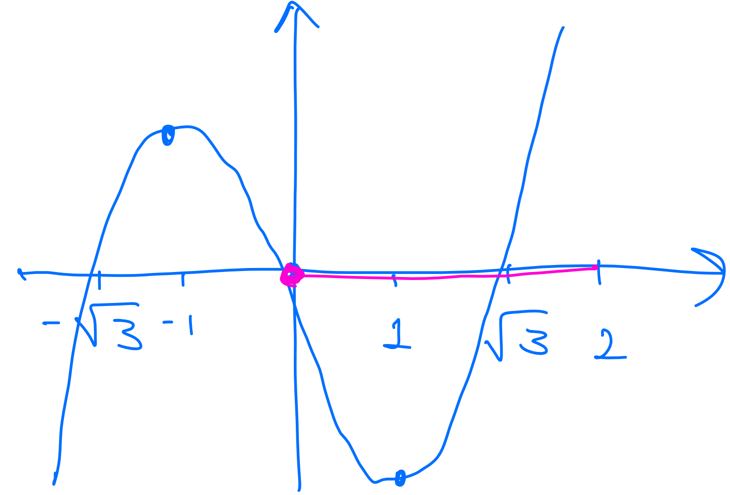
- Define decision variables
- Express objective function and constraints using the decision variables
- Solve the problem (easy to say….)
  - Graphical method
  - Simplex
  - Excel (Solver)
  - Lingo
  - CPLEX or other optimization software
  - …and more

Applications

diet problem

- Blending petroleum products: maximise the profit subject to constraints on quantity of components of petroleum available (LP).
- Portfolio design: maximise the expected return subject to constraints on maximum levels of risk acceptable (NLP).
- Aircrew scheduling: minimise the cost to an airline (wages plus accommodation expenses, etc) subject to all flights having the required crew, all crew returning to their home bases, and all union and legal requirements on work schedules met (IP).
- What do you think?

Complications



Consider the problem:

$$max\ f(x) = x^3 - 3x$$

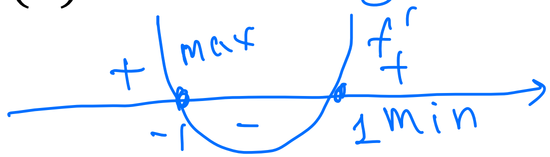
s.t. $\quad x \leq 2$

$$x \geq 0$$

Feasible region $0 \leq x \leq 2$

The feasible region is , the end points are $x = 0$ and $x = 2$

$f'(x) = 3x^2 - 3$; the stationary points are $x = -1$ and $x = 1$

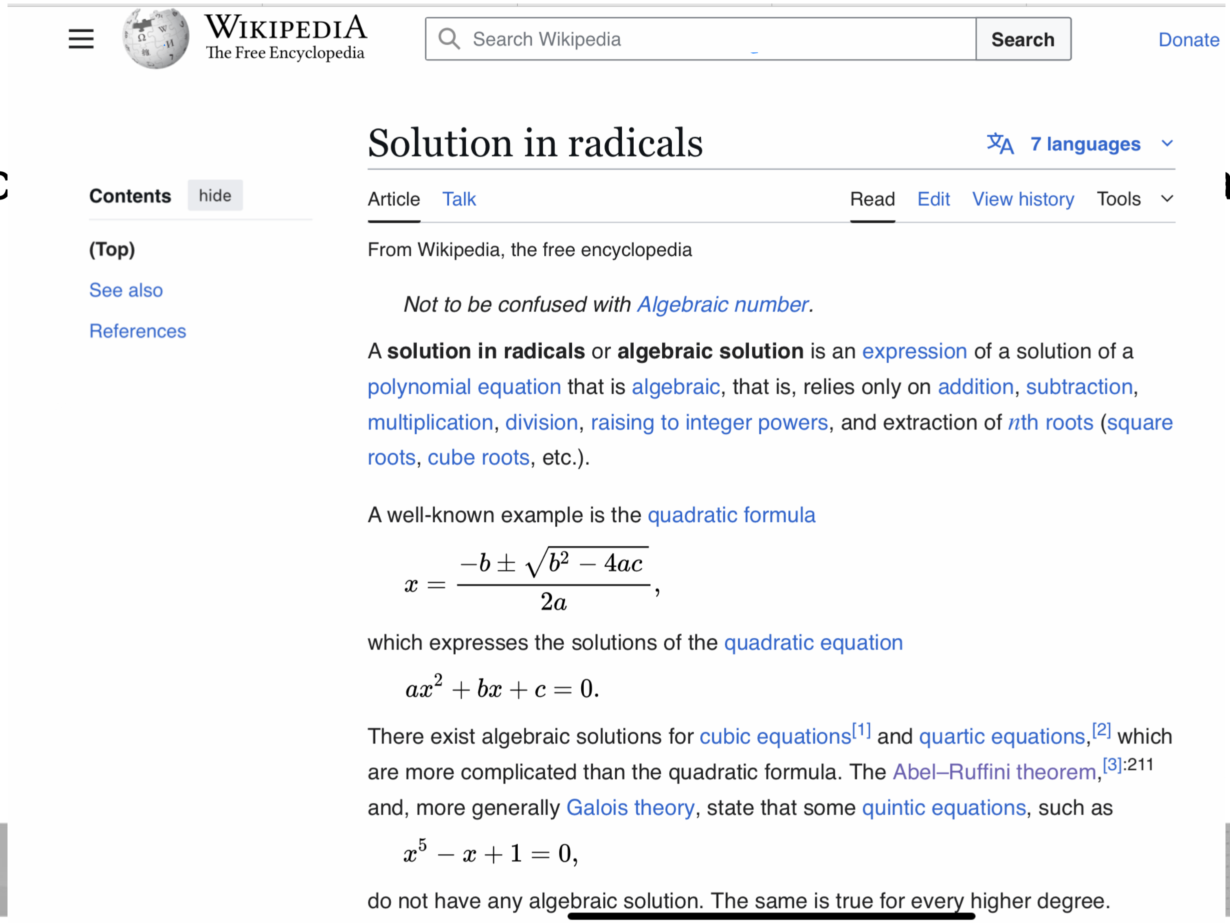


$max\ f(x) = \max\{ f(0), f(1), f(2) \}$ - 3 values

Complications

- The Abel–Ruffini theorem states that there is no solution in radicals to general polynomial equations of degree five or higher with arbitrary coefficients.

- Much more c ıbles)



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References

Solution in radicals

7 languages

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Not to be confused with Algebraic number.

A **solution in radicals** or **algebraic solution** is an expression of a solution of a polynomial equation that is algebraic, that is, relies only on addition, subtraction, multiplication, division, raising to integer powers, and extraction of nth roots (square roots, cube roots, etc.).

A well-known example is the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

which expresses the solutions of the quadratic equation

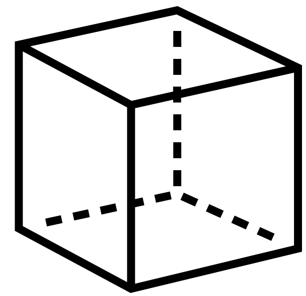
$$ax^2 + bx + c = 0.$$

There exist algebraic solutions for cubic equations[1] and quartic equations,[2] which are more complicated than the quadratic formula. The Abel–Ruffini theorem,[3]:211 and, more generally Galois theory, state that some quintic equations, such as

$$x^5 - x + 1 = 0,$$

do not have any algebraic solution. The same is true for every higher degree.

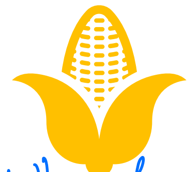
Complications



- How many extreme points in a square? $4 = 2^2$
- How many extreme points in a cube? 2^3
- How many extreme points in $n - dimensional\ \ hypercube$? 2^n
- How to find a solution without full enumeration of extreme points?

Farmer Jones example…

Let ***W*** – be the number of acres under  and

C – be the number of acres under 

OF: max Revenue

$$\max z = \overbrace{\underbrace{W}_{\text{acre}} \times \underbrace{2\left(\frac{\cancel{tonnes}}{\cancel{acre}}\right)}_{\text{yield}} \times 8\left(\frac{\$}{\cancel{tonne}}\right)}^{\text{rev. from wheat}} + \overbrace{C \times 3\left(\frac{tonnes}{acre}\right) \times 5\left(\frac{\$}{tonne}\right)}^{\text{revenue from corn}} = \underline{16W + 15C}$$

s.t $\quad W + C \le 4 \quad \rightarrow$ capacity N of aches

$$\underbrace{W}_{\text{acres}} \times 2\left(\frac{tonnes}{\cancel{acre}}\right) \le 4 \leftrightarrow \boxed{W \le 2}$$

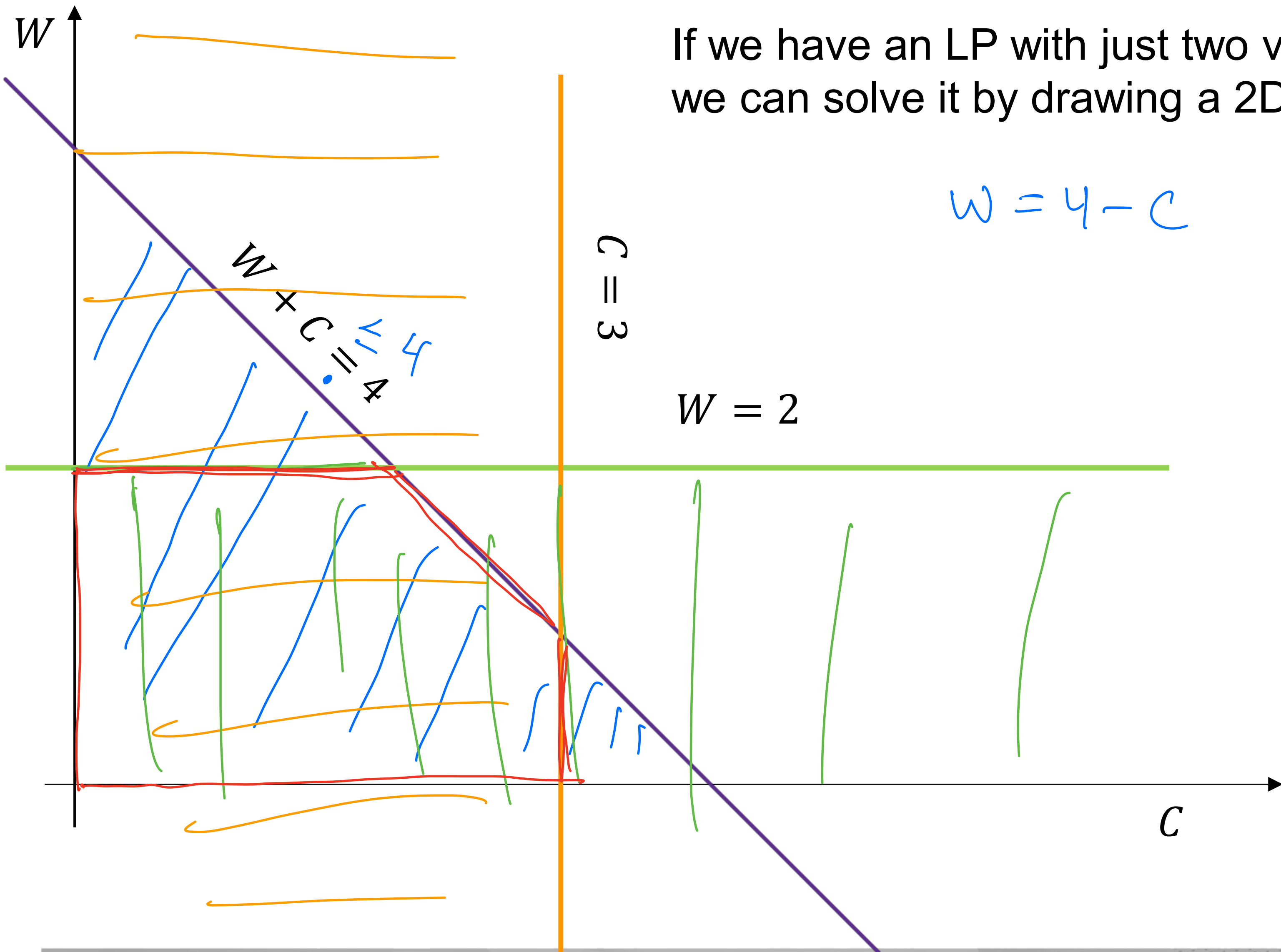
$$C \times 3\left(\frac{tonnes}{acre}\right) \le 9 \leftrightarrow \boxed{C \le 3}$$

max selling TONNES

$$W, C \ge 0$$

• graphical method

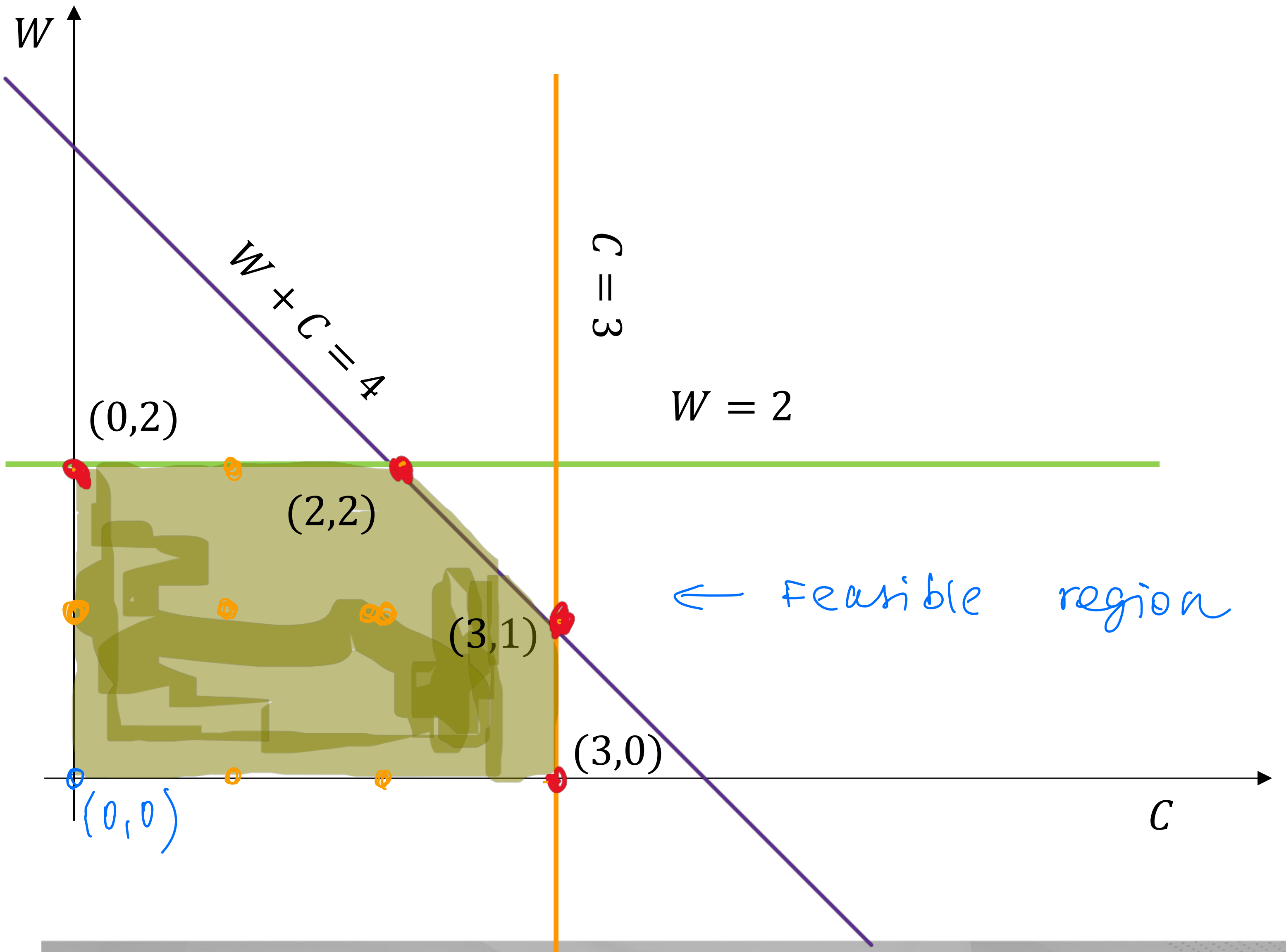
Farmer Jones example…



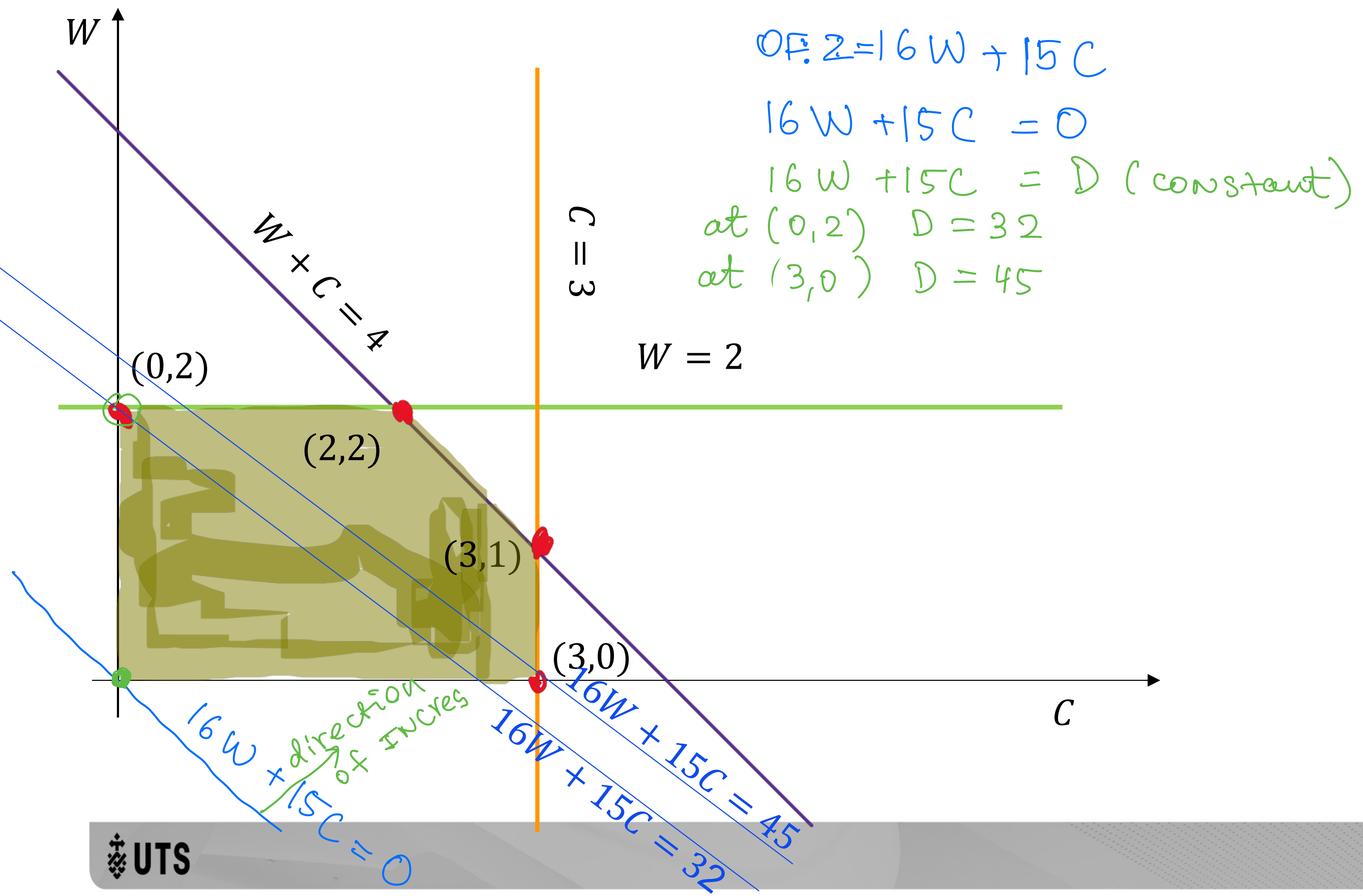
If we have an LP with just two variables, we can solve it by drawing a 2D diagram.

$W = 4 - C$

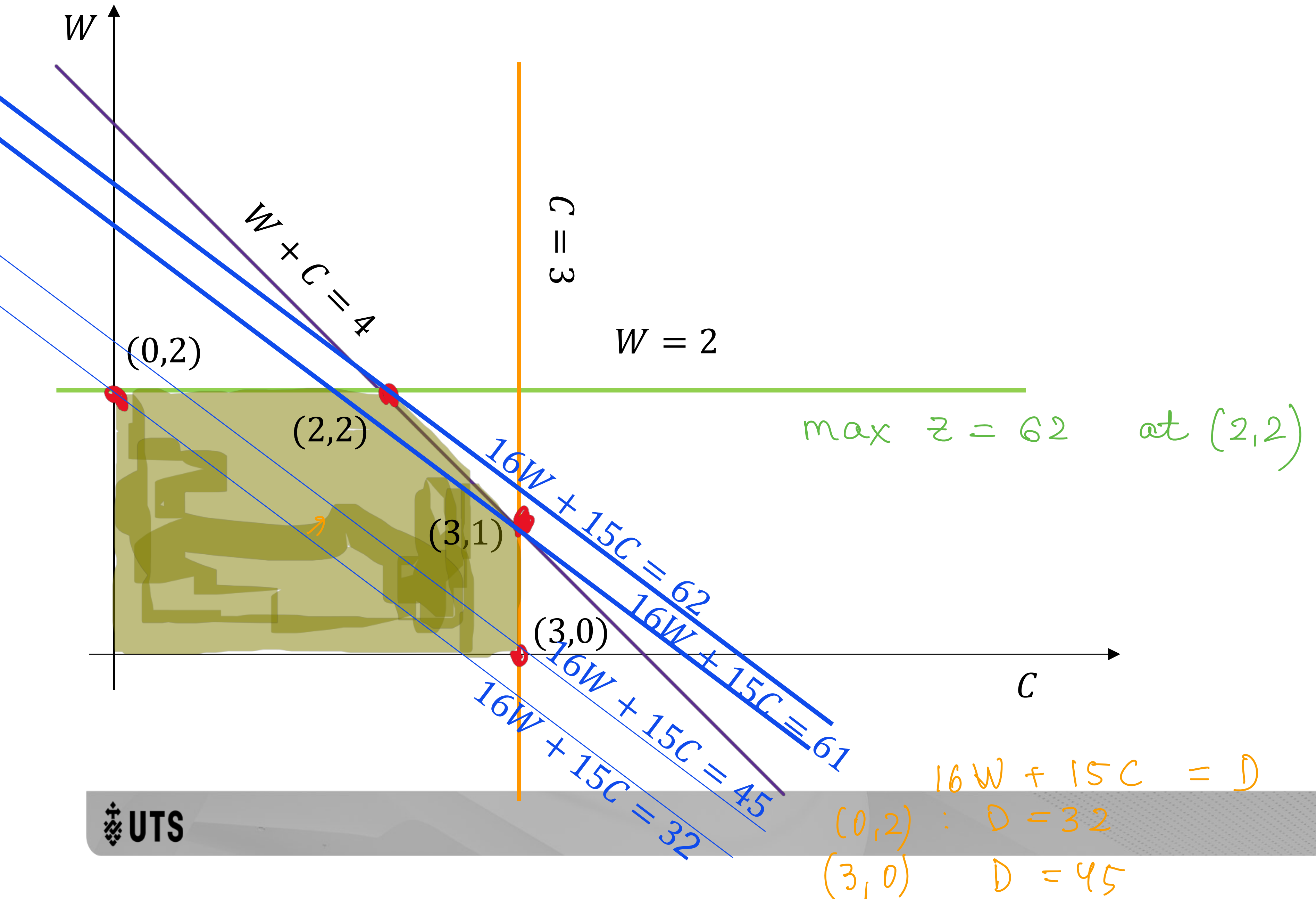
Farmer Jones example…



Farmer Jones example…



Farmer Jones example…



Assessment :

Test 1 :	15%	45 min	IN class
Test 2	25%	50 min	
Final exam	30%	- online	
Modelling Written Report	30%		

(2, 2) D = 62
(3, 1) D = 15 x 3 + 16 = 61

Blending problem

A one-kilogram pack of dog food must contain protein (at least 31%), carbohydrate (at least 38%), and fat (between 13% and 15%, that is greater than or equal to 13% and less than or equal to 15%). Three foods (food1, food2 and food3) are to be blended together in various proportions to produce a least-cost pack of dog food satisfying these requirements. The information for one kilogram of each food is given in the table below.

	protein *(grams)*	*carbohydrate* *(grams)*	*fat* *(grams)*	*price* *(dollars)* per kg
food1	300	500	90	3
food2	450	300	160	1
food3	200	400	200	2

Let x_1 x_2 x_3 be the amount of food 1 food 2 food 3, IN kg in the mix

Let x_i be the amount of food i IN the mix $(i = 1, 2, 3)$ in kg.

$$\min \quad 3x_1 + 1 \cdot x_2 + 2x_3$$

s.t.

$$x_1 + x_2 + x_3 = 1.$$

(1 kg of food)

protein $\geq 31\%$

carbs $\geq 38\%$

$13\% \leq$ fat $\leq 15\%$

- $\underbrace{0.3x_1 + 0.45x_2 + 0.2x_3}_{\text{amount of protein in the mix}} \geq 0.31$

(0.31 kg is 31% of 1 kg)

- $0.5x_1 + 0.3x_2 + 0.4x_3 \geq 0.38 \rightarrow$ carbs
- $\underbrace{0.09x_1 + 0.16x_2 + 0.2x_3}_{\text{fat in mix}} \geq 0.13$
- $0.09x_1 + 0.16x_2 + 0.2x_3 \leq 0.15$

$$x_1, x_2, x_3 \geq 0$$

	protein (grams)	carbohydrate (grams)	fat (grams)
d1	300 0.3	500	90
d2	450 0.45	300	160
d3	200 0.2	400	200

Formulation with „set" notation

$j = 1, 2, 3$

	protein (grams)	carbohydrate (grams)	fat (grams)	price (dollars) selling price per kg
food1	300 0.3	500	90	3
food2	450 0.45	300	160	1
food3	200 0.2	400	200	2

$i = 1, 2, 3$

OF ?

DV: x_i - amount of food i in the mix (in kg).

Parameters: set of „Intgredients": {protein, carbs, fat}

$j = 1, 2, 3$

Let Ing_{ij} - amount of Ingredient j in food i (in kg)

$i = 1, 2, 3$
$j = 1, 2, 3$

$ING_{2,3} = 0.16$

$ING_{3,1} = 0.2$

$SellPrice_i$ → selling price of food i, per kg

Let LB_j - be the min amount of ingredient j in the mix, $j = 1, 2, 3$

UB_j - be the max amount of ingredient j in the mix

	protein (grams)	carbohydrate (grams)	fat (grams)	price (dollars)
food1	300	500	90	3
food2	450	300	160	1
food3	200	400	200	2
LB, kg	0.31	0.38	0.13	
UB, kg	1 - 0.38 - 0.13	1 - 0.13 - 0.31	0.15	

OF : $\min \sum_{i=1}^{3} \text{selling\_price}_i \times x_i$

s.t. $\sum_{i=1}^{3} x_i = 1$ - 1 kg

For each Ingredient $j = 1, 2, 3$

$$\sum_{i=1}^{3} ING_{ij} \times x_i \geq LB_j$$

$$\sum_{i=1}^{3} ING_{ij} \times x_i \leq UB_j$$

$$x_i \geq 0$$

Blending problem 2

A company should produce steel from the following three alloys:

	$cost_i$ cost (per tonne)	ING_{ij} carbon	chrome	silicon	$Av._i$ available (tonne)
Alloy 1	\$52	5%	0.1%	3%	~~unlimited~~ M
Alloy 2	\$71	3.8%	0.4%	2%	2000
Alloy 3	\$46	2%	0.3%	2.8%	5000

The chemical content of a blend is the weighted average of the chemical content of its components. For example, the portion of carbon in a blend of 20 tonnes of Alloy 1 and 30 tonnes of Alloy 2 is

$$\frac{20 \times 0.05 + 30 \times 0.038}{(20+30)} = 0.0428$$

$$\frac{20 \times 0.05 + 30 \times 0.038}{20 + 30} = 0.0428,$$

that is 4.28%.

The steel should satisfy the following quality requirements

	at least LB_j	not more than UB_j
carbon	3.5%	4 %
chrome	0.2%	0.5 %
silicon	2.7%	2.7 %

and will be sold for \$65 per tonne.

The company wishes to maximise its profit.

DV: x_i - amount of TONNES alloy i used

$cost_i$ - cost of buying alloy i, $i=1,2,3$

$Available_i$ - amount of alloy i available [assume

ING_{ij} - amount of ingredient j in alloy i, in %

ING_{33} - 0.028

LB_j, UB_j → upper and lower bounds for ING j, in %

selling-p = \$65 per TONN

OF: max $\underbrace{\text{selling-p} \times \sum_{i=1}^{3} x_i}_{\text{revenue}} - \underbrace{\sum_{i=1}^{3} cost_i \times x_i}_{\text{production costs}}$

s.t.

Capacity constraint : $x_i \leq Available_i$, $i=1,2,3$

For each ingredient j

UB: $\sum_{i=1}^{3} ING_{ij} \times x_i \leq UB_j \times \underbrace{\sum_{i=1}^{3} x_i}_{\text{Total amount of steel}}$

% × amount   %

LB: $\sum_{i=1}^{3} ING_{ij} \times x_i \geq LB_j \times \sum_{i=1}^{3} x_i$

$x_i \geq 0$

Production planning

Giapetto's wooden soldiers and trains. Each soldier sells for $27, uses $10 of raw materials and takes $14 of labor & overhead costs. Each train sells for $21, uses $9 of raw materials, and takes $10 of overhead costs. Each soldier needs 2 hours finishing and 1 hour carpentry; each train needs 1 hour finishing and 1 hour carpentry. Raw materials are unlimited, but only 100 hours of finishing and 80 hours of carpentry are available each week. Demand for trains is unlimited; but at most 40 soldiers can be sold each week. How many of each toy should be made each week to maximize profits?

(Winston 3.1, p. 49)

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Let x_i be the number of toy type i made: $i=1$ - soldiers
$i=2$ - trains

	DV	$sellP_i$ \$	raw_i \$	$cost_i$ \$	Labour fin_i	car_i	Demand
Sold.	x_1	\$27	\$10	\$14	2	1	40
Train	x_2	\$21	\$9	\$10	1	1	M
				Available	100	80	

OF: $\sum_{i=1}^{2} x_i \times (selling\text{-}p_i - raw_i - cost_i)$

s.t.

$2x_1 + 1x_2 \leq 100$ → finishing time

$x_1 + x_2 \leq 80$ → carpentry time

$x_1 \leq 40$ → demand soldiers

$x_i \geq 0$ and <u>Integer</u>!

* Integer programming modelling: Portfolio selection problem

HW

Over a 3-year planning horizon, an investor can invest in the following projects:

project	investment			return
	Year 1	Year 2	Year 3	
Project 1	2	8	4	35
Project 2	5	4	10	22
Project 3	9	3	13	39
Project 4	1	5	7	30
Project 5	11	6	3	31
Project 6	4	4	6	19
Project 7	2	12	4	20

The following funds are available over this 3-year planning horizon:

	Year 1	Year 2	Year 3
available funds	33	40	41

Portfolio selection problem

The investment strategy should satisfy the following requirements:

- the investor can invest in at most four projects;
- if the investor invests in Project 1, then he can invest in at most two of the five projects - Project 2, Project 3, Project 4, Project 5 and Project 7;
- if the investor invests in Project 2, then he must also invest in at least two of the three projects - Project 3, Project 4 and Project 6;
- if the investor invests in either Project 5 or Project 6 or in both, then he must invest in at least one of the two projects - Project 2 or Project 4.
- if the investor invests in Project 1, then he cannot invest in Project 6;
- if the investor invests in one of the two projects - Project 1 or Project 3, then he must also invest in the other one;

The investor wishes to maximise the total return from his investments.

Integer programming modelling: workforce planning

A company has only full-time employees. The following table specifies the number of personnel required each day:

	required number
Monday	20
Tuesday	13
Wednesday	10
Thursday	12
Friday	15
Saturday	9
Sunday	17

If each employee works five days per week, what is the minimum number of employees needed in order to meet the daily requirements?

Modeling: a summary

The steps involved in the formulation of a mathematical optimisation model are:

a. Identify the decision variables (whose quantities can be controlled by the decision maker).

b. Construct the objective function and constraints (restrictions) in terms of the decision variables. (You should not introduce new variables (unknowns) at this point.

c. Ask yourself whether there is any hidden assumption that may exist,

e.g. integer decision variables, fractional decision variables.
The most common "hidden" assumption is that the variables are nonnegative.

d. Write the complete formulation as a linear program, objective first, followed underneath by the constraints.

e. It is usually a good idea to write the formulation out first before constructing the spreadsheet until you get experience at formulation and structuring spreadsheets.