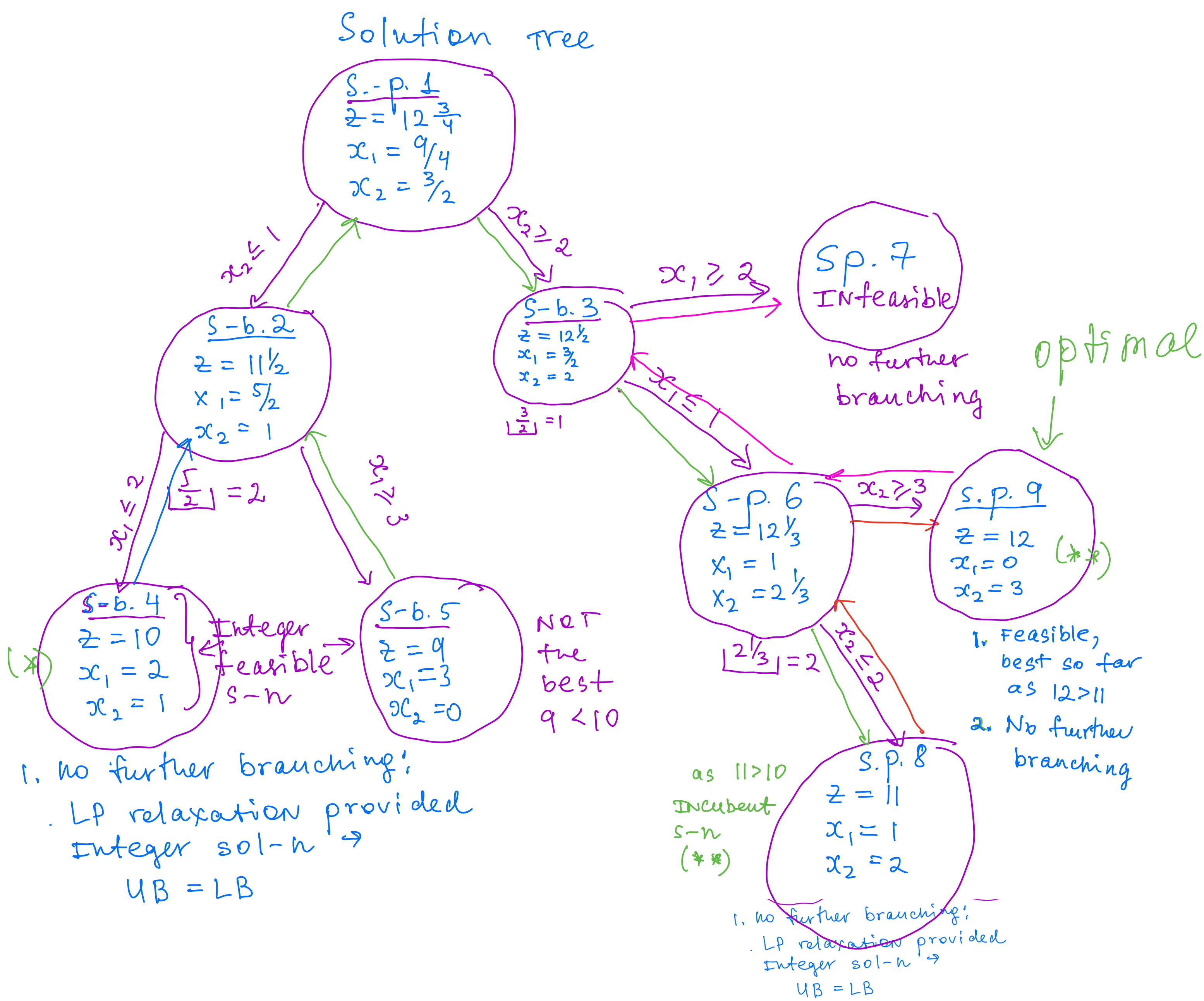
37242 Introduction to Optimisation

Tutorial 11

1. Solve the following integer program

$$
\begin{array}{rl}
\max & z = f(\mathbf{x}) = 3x_1 + 4x_2 \\
\text{s.t.} & 2x_1 + x_2 \leq 6 \\
 & 2x_1 + 3x_2 \leq 9 \\
\text{with} & x_1, x_2 \text{ nonnegative and integral.}
\end{array}
$$



1. Solve the following integer program using Branch and Bound Algorithm.

$$\begin{array}{rlcl} \max \ z = & 3x_1 + 4x_2 & & \\ s.t. & 2x_1 + x_2 & \leq & 6 \\ & 2x_1 + 3x_2 & \leq & 9 \\ x_1, x_2 & \text{integers and} & \geq & 0 \end{array}$$

S-p. 1 - solve without integrality constraint

$$\begin{array}{rlcl} \max \ z = & 3x_1 + 4x_2 & & \\ s.t. & 2x_1 + x_2 & \leq & 6 \ + S_1 \\ & 2x_1 + 3x_2 & \leq & 9 \ + S_2 \\ x_1, x_2 & \cancel{\text{integers and}} & \geq & 0 \end{array}$$

LP relaxation

$x_B = (S_1 S_2)$

	X_1	X_2	S_1	S_2	RHS	
z	-3	-4	0	0	0	
S_1	2	1	1	0	6	
S_2	2	3	0	1	9	
z	$-1/3$	0	0	$4/3$	12	
S_1	$4/3$	0	1	$-1/3$	3	$9/4$
x_2	$2/3$	1	0	$1/3$	3	$9/2$
z	0	0	$1/4$	$5/4$	$12\frac{3}{4}$	
x_1	1	0	$3/4$	$-1/4$	$9/4$	
x_2	0	1	$-1/2$	$1/2$	$3/2$	

Optimal for LP relaxtion, but Not feasible for IP

↓

branch on x_1 or x_2

$x_2 = 3/2$

$\lfloor \frac{3}{2} \rfloor = 1$

$x_2 \leq 1$ → S-p. 2

$x_2 \geq 2$ → S.-p. 3

S.-P. 2

To solve S.-P. 2 use optimal tablaeu of S.-P. 1 (parent node) and add the new constraint $x_2 \leq 1$

$x_2 \leq 1 \rightarrow x_2 + S_3 = 1$ as $x_1 = 9/4$, $x_2 = 3/2$ does not satisfy the new constraint, S_3 must enter the basis.

as $x_2 - \frac{1}{2} S_1 + \frac{1}{2} S_2 = 3/2$ (from the Tableau)

$S_3 + \underbrace{3/2 + 1/2\, S_1 - 1/2\, S_2}_{x_2} = 1 \rightarrow 1/2\, S_1 - 1/2\, S_2 + S_3 = -1/2 \rightarrow$ canonical form

	x_1	x_2	S_1	S_2	S_3	
z	0	0	1/4	5/4	0	$12\frac{3}{4}$
x_1	1	0	3/4	-1/4	0	9/4
x_2	0	1	-1/2	1/2	0	3/2
S_3	0	0	1/2	-1/2	1	-1/2

Initial Tableau for S.-P. 2.

Basis	x_1	x_2	s_1	s_2	s_3	s_4	s_5	rhs
Subproblem 1								
z	-3	-4	0	0				0
s_1	2	1	1	0				6
s_2	2	$\boxed{3}$	0	1				9
z	$-\frac{1}{3}$	0	0	$\frac{4}{3}$				12
s_1	$\frac{4}{3}$	0	1	$-\frac{1}{3}$				3
x_2	$\frac{2}{3}$	1	0	$\frac{1}{3}$				3
z	0	0	$\frac{1}{4}$	$\frac{5}{4}$				$12\frac{3}{4}$
x_1	1	0	$\frac{3}{4}$	$-\frac{1}{4}$				$\frac{9}{4}$
x_2	0	1	$-\frac{1}{2}$	$\frac{1}{2}$				$\frac{3}{2}$
Subproblem 2								
$x_2 \le 1$	x_1	x_2	s_1	s_2	s_3			
z	0	0	$\frac{1}{4}$	$\frac{5}{4}$	0			$12\frac{3}{4}$
x_1	1	0	$\frac{3}{4}$	$-\frac{1}{4}$	0			$\frac{9}{4}$
x_2	0	1	$-\frac{1}{2}$	$\frac{1}{2}$	0			$\frac{3}{2}$
s_3	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	1			$-\frac{1}{2}$
z	0	0	$\frac{3}{2}$	0	$\frac{5}{2}$			$11\frac{1}{2}$
x_1	1	0	$\frac{1}{2}$	0	$-\frac{1}{2}$			$\frac{5}{2}$
x_2	0	1	0	0	1			1
s_2	0	0	-1	1	-2			1
Subproblem 4								
$x_1 \le 2$	x_1	x_2	s_1	s_2	s_3	s_4		
z	0	0	$\frac{3}{2}$	0	$\frac{5}{2}$	0		$11\frac{1}{2}$
x_1	1	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	0		$\frac{10}{4}$
x_2	0	1	0	0	1	0		1
s_2	0	0	-1	1	-2	0		1
s_4	0	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	1		$-\frac{1}{2}$
z	0	0	0	0	4	3		10
x_1	1	0	0	0	0	1		2
x_2	0	1	0	0	1	0		1
s_2	0	0	0	1	-3	-2		2
s_1	0	0	1	0	-1	-2		1

$x_1 + s_4 = 2 \rightarrow$ bring to canonical form

$\underbrace{\frac{10}{4} - \frac{1}{2}s_1 + \frac{1}{2}s_3}_{x_1} + s_4 = 2 \rightarrow -\frac{1}{2}s_1 + \frac{1}{2}s_3 + s_4 = -\frac{1}{2}$

Initial Tableau for S.P. 4

With the solution of the subproblem 4, we have a candidate for the optimal solution, i.e. lower bound, $x_1^* = 2$, $x_2^* = 1$ and $LB = 10$.

new variable entered basis

↓ optimal Tableau of s-b. 2

Basis	x_1	x_2	s_1	s_2	s_3	s_4	s_5	rhs
Subproblem 5								
$x_1 \geq 3$								
z	0	0	$\frac{3}{2}$	0	$\frac{5}{2}$		0	$11\frac{1}{2}$
x_1	1	0	$\frac{1}{2}$	0	$-\frac{1}{2}$		0	$\frac{10}{4}$
x_2	0	1	0	0	1		0	1
s_2	0	0	-1	1	-2		0	1
s_5	0	0	$\frac{1}{2}$	0	$-\frac{1}{2}$		1	$-\frac{1}{2}$
z	0	0	4	0	0		5	9
x_1	1	0	0	0	0		-1	3
x_2	0	1	1	0	0		2	0
s_2	0	0	-3	1	0		-4	3
s_3	0	0	-1	0	1		-2	1

The feasible solution obtained from the subproblem 5 is not better that the current lower bound $LB = 10$, so subproblem 5 shall be fathomed.
All possibilities on the first side of the first branch have now been solved or fathomed. Then we return to the second side of the first branch.

Opt. tableau for s.p. 1

Basis	x_1	x_2	s_1	s_2	s_6	s_7	s_8	rhs
z	0	0	$\frac{1}{4}$	$\frac{5}{4}$				$12\frac{3}{4}$
x_1	1	0	$\frac{3}{4}$	$-\frac{1}{4}$				$\frac{9}{4}$
x_2	0	1	$-\frac{1}{2}$	$\frac{1}{2}$				$\frac{3}{2}$
Subproblem 3	$x_2 - s_6 = 2 \to$ bring to canonical for							
$x_2 \geq 2$	x_1	x_2	s_1	s_2	s_6			
z	0	0	$\frac{1}{4}$	$\frac{5}{4}$	0			$12\frac{3}{4}$
x_1	1	0	$\frac{3}{4}$	$-\frac{1}{4}$	0			$\frac{9}{4}$
x_2	0	1	$-\frac{1}{2}$	$\frac{1}{2}$	0			$\frac{3}{2}$
s_6	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	1			$-\frac{1}{2}$
z	0	0	0	$\frac{3}{2}$	$\frac{1}{2}$			$12\frac{1}{2}$
x_1	1	0	0	$\frac{1}{2}$	$\frac{3}{2}$			$\frac{3}{2}$
x_2	0	1	0	0	-1			2
s_1	0	0	1	-1	-2			1
Subproblem 6								
$x_1 \leq 1$	$x_1 + s_7 = 1$							
z	0	0	0	$\frac{3}{2}$	$\frac{1}{2}$	0		$12\frac{1}{2}$
x_1	1	0	0	$\frac{1}{2}$	$\frac{3}{2}$	0		$\frac{3}{2}$
x_2	0	1	0	0	-1	0		2
s_1	0	0	1	-1	-2	0		1
s_7	0	0	0	$-\frac{1}{2}$	$-\frac{3}{2}$	1		$-\frac{1}{2}$
z	0	0	0	$\frac{4}{3}$	0	$\frac{1}{3}$		$12\frac{1}{3}$
x_1	1	0	0	0	0	1		1
x_2	0	1	0	$\frac{1}{3}$	0	$-\frac{2}{3}$		$2\frac{1}{3}$
s_1	0	0	1	$-\frac{1}{3}$	0	$-\frac{4}{3}$		$1\frac{2}{3}$
s_6	0	0	0	$\frac{1}{3}$	1	$-\frac{2}{3}$		$\frac{1}{3}$

Hence, we branch on x_2 again, giving either $x_2 \leq 2$ (Subproblem 8) with $x_2^* = 2, x_1^* = 1, z^* = 11$ or $x_2 \geq 3$ (Subproblem 9) with $x_2^* = 3, x_1^* = 0, z^* = 12$. So our LB was updated from 10 to 11, and then from 11 to 12. Finally, we need to consider $x_1 \geq 2$ (Subproblem 7) from the second side, second branch.

Basis	x_1	x_2	s_1	s_2	s_6	s_7	s_8	rhs
Subproblem 7								
$x_1 \geq 2$	$x_1 - s_8 = 2$							
z	0	0	0	$\frac{3}{2}$	$\frac{1}{2}$		0	$12\frac{1}{2}$
x_1	1	0	0	$\frac{1}{2}$	$\frac{3}{2}$		0	$\frac{3}{2}$
x_2	0	1	0	0	-1		0	2
s_1	0	0	1	-1	-2		0	1
s_8	0	0	0	$\frac{1}{2}$	$\frac{3}{2}$		1	$-\frac{1}{2}$ < 0 leaving

as there are no $\hat{a}_{4j} < 0 \rightarrow$ Infeasibl

As there is no negative denominator in the ratio for the dual simplex method, this subproblem is infeasible and shall be fathomed. All the subproblems in our branching tree are solved or fathomed, so we can claim that our current LB=12, which comes from the candidate optimal solution $x_1^* = 0, x_2^* = 3$, $z^* = 12$, is the optimal solution.