37242 Introduction to Optimisation

Tutorial 6

1. Consider the LP

$\min z = x_1 + x_2$

s.t. $2x_1 - x_2 \geq 6$

$x_1 + 3x_2 \leq 10$

$x_1 - x_2 \geq \frac{3}{2}$

$x_1,\ x_2 \geq 0$

Solve the dual problem and, with the Complementary slackness Theorem, obtain the solution for the primal problem.

2. Use dual Simplex method to solve:

$\min\ z = 5x_1 + 2x_2 + 8x_3$

s.t. $2x_1 - 3x_2 + 2x_3 \geq 3$

$-x_1 + x_2 + x_3 \geq 5$

$x_1,\ x_2,\ x_3 \geq 0$

* from Linear and Non-Linear Programming by S.G.Nash and A.Sofer

Strong duality theorem

Let x be a feasible solution for primal LP (P) and y be a feasible solution of the corresponding dual LP (D). Then

$$\boldsymbol{c}^T\boldsymbol{x} = \boldsymbol{b}^T\boldsymbol{y}$$

if and only if x is an optimal solution for (P) and y is an optimal solution for (D).

(We need the dual of the LP in the standard form)

Corollary

If an optimal solution to the dual LP (D) is obtained, an optimal solution to its primal LP (P) can be readily obtained and both optimal objective values are equal.

Complementary slackness theorem

Theorem 6

- Let x be a feasible solution to the primal LP (P) and y be a feasible solution to the dual LP (D). Both solutions x and y are optimal to the primal (P) and dual (D), respectively, if and only if they satisfy

$$(\boldsymbol{c}^T - \boldsymbol{y}^T A)\boldsymbol{x} = \boldsymbol{0} \quad \textit{and} \quad \boldsymbol{y}^T(\boldsymbol{b} - A\boldsymbol{x}) = \boldsymbol{0} \qquad (2)$$

THAT is:

for each i^{th} dual constraint:

$$(c_i - y^T A_i) x_i = 0$$

IF in opt x, $x_i \neq 0$

$\downarrow$

$c_i - y^T A_i = 0$

(active constraint)

and

for each j^{th} primal constraint:

$$y_j (b_j - A_j x) = 0.$$

if in opt y, $y_j \neq 0$

$\downarrow$

$b_j - A_j x = 0$

(active constraint)

binding

1. Consider the LP

$$\min z = x_1 + x_2$$

s.t.

$2x_1 - x_2 \geq 6$ • normal (P) form $\rightarrow y_1 \geq 0$

$x_1 + 3x_2 \leq 10$ • reverse to normal form $\rightarrow y_2 \leq 0$

$x_1 - x_2 \geq \frac{3}{2}$ • normal form $\rightarrow y_3 \geq 0$

$x_1,\ x_2 \geq 0$ $\rightarrow$ normal form $\rightarrow$ dual const. in normal form

Solve the dual problem and, with the Complementary slackness Theorem, obtain the solution for the primal problem.

Assymetrical Dual:

$$\max\ w = 6y_1 + 10y_2 + \tfrac{3}{2}y_3$$

s.t.

$$2y_1 + y_2 + y_3 \leq 1$$

$$-y_1 + 3y_2 - y_3 \leq 1$$

$$y_1, y_3 \geq 0,\quad y_2 \leq 0$$

$$\downarrow$$

$$y_2' = -y_2 \rightarrow y_2' \geq 0$$

Standard form:

$$\max\ w = 6y_1 - 10y_2' + \tfrac{3}{2}y_3$$

s.t.

$$2y_1 - y_2' + y_3 + s_1 = 1$$

$$-y_1 - 3y_2' - y_3 + s_2 = 1$$

$$y_1\ y_2'\ y_3\ s_1\ s_2 \geq 0.$$

$X_{B_0} = (s_1, s_2)$

max problem!

	y_1	y_2'	y_3	s_1	s_2	RHS	
w	-6	10	$-3/2$	0	0	0	
s_1	$\boxed{2}$	-1	1	1	0	1 $\rightarrow$	
s_2	$-1 \rightarrow 0$	-3	-1	0	1	1	
				$c_B^T B^{-1}$			
w	0	7	$3/2$	3	0	3	$R_0' = R_0 + 6R_1'$
y_1	1	$-1/2$	$1/2$	$1/2$	0	$1/2$	$R_1' = R_1/2$
s_2	0	$-7/2$	$-1/2$	$1/2$	1	$3/2$	$R_2' = R_2 + R_1'$

optimal $\hat{c}_N \geq 0$

$w^* = 3$

$y = \begin{pmatrix} 1/2 \\ 0 \\ 0 \end{pmatrix}$ } dual opt. solution and OF value

↓

$z^* = w^* = 3$ by Strong Duality Theorem

↓

$x_1^* + x_2^* = 3$

as $y_1 \neq 0 \rightarrow 1^{st}$ (P) constraint is active "="

by comp. slackness Theorem

↓

$2x_1^* - x_2^* = 6$

↓

$$\begin{cases} x_1^* + x_2^* = 3 \\ 2x_1^* - x_2^* = 6 \end{cases} +$$

$3x_1^* = 9 \rightarrow x_1^* = 3$

$x_2^* = 0$

2. Use dual Simplex method to solve:

min $z = 5x_1 + 2x_2 + 8x_3$

s.t. $2x_1 - 3x_2 + 2x_3 \geq 3$

$-x_1 + x_2 + x_3 \geq 5$

$x_1, x_2, x_3 \geq 0$

(P) → (D)

max $3y_1 + 5y_2$

s.t $2y_1 - y_2 \leq 5$

$-3y_1 + y_2 \leq 2$

$2y_1 + y_2 \leq 8$

$y_1 y_2 \geq 0$

* from Linear and Non-Linear Programming by S.G.Nash and A.Sofer

Standard form ( ).

min $z = 5x_1 + 2x_2 + 8x_3$

s.t. $2x_1 - 3x_2 + 2x_3 - e_1 = 3$

$-x_1 + x_2 + x_3 - e_2 = 5$

$x_1 \; x_2 \; x_3 \; e_1 \; e_2 \geq 0.$

not canonical $\times(-1)$

$x_B = (e_1 e_2)$

↓

$-2x_1 + 3x_2 - 2x_3 + e_1 = -3$

$x_1 - x_2 - x_3 + e_2 = -5$

$x_1 x_2 x_3 e_1 e_2 \geq 0.$

min $z = 5x_1 + 2x_2 + 8x_3$

$z - 5x_1 - 2x_2 - 8x_3 = 0$

or

$\hat{c}_N^T = \underbrace{c_{B_0}^T B_0^{-1}}_{} N - c^T = -c^T = (-5, -2, -8)$

$z(x_{B_0}) = c_B^T B^{-1} b = 0$ $(0, 0)$

	x_1	x_2	x_3	e_1	e_2	RHS	
z	-5	-2	-8	0	0	0	
e_1	-2	3	-2	1	0	$-3<0$	①
e_2	1	$\boxed{-1}$	-1	0	1	$-5<0$	
z	-7	0	-6	0	-2	10	
e_1	1	0	$\boxed{-5}$	1	3	$-18<0$	②
x_2	-1	1	1	0	-1	5	
z	$-\frac{41}{5}$	0	0	$-\frac{6}{5}$	$-\frac{28}{5}$	$\frac{158}{5}$	
x_3	$-\frac{1}{5}$	0	1	$-\frac{1}{5}$	$-\frac{3}{5}$	$\frac{18}{5}$	③
x_2	$-\frac{4}{5}$	1	0	$\frac{1}{5}$	$-\frac{2}{5}$	$\frac{7}{5}$	

① Optimal, not Feasible

Ratio Test:

$$\left|\frac{-2}{-1}\right| < \left|\frac{-8}{-1}\right|$$

e_2 leaving as $-5 < -3$

x_2 enters

$R_0' = R_0 + 2R_2'$

$R_1' = R_1 - 3R_2'$

$R_2' = R_2 \times (-1)$

② Optimal, not Feasible

e_1 leaves

$R_0' = R_0 + 6R_1'$

$R_1' = \frac{R_1}{(-5)}$

$R_2' = R_2 - R_1'$

Optimal as $\hat{c}_N < 0$

$$z^* = \frac{158}{5} = 31.6$$

$$x = \begin{pmatrix} 0 \\ 7/5 \\ 18/5 \end{pmatrix} \Big\} \neq 0 \rightarrow 2^{nd}, 3^{rd}$$ dual const. are active

Dual

$$\max \quad 3y_1 + 5y_2$$

s.t.

$$2y_1 - y_2 \le 5$$

$$-3y_1 + y_2 \le 2$$

$$2y_1 + y_2 \le 8$$

$$y_1, y_2 \ge 0$$

$$w^* = z^* = \frac{158}{5} \rightarrow \begin{cases} 3y_1^* + 5y_2^* = \frac{15}{5} \\ -3y_1^* + y_2^* = 2 \end{cases} +$$

$$6y_2^* = \frac{168}{5} \rightarrow y_2^* = \frac{168}{6 \times 5} = \frac{28}{5}$$

$$y_1^* = \frac{6}{5}$$