37242 Introduction to Optimisation

Tutorial 11

1. Solve the following integer program

$$\begin{aligned} \text{max} \quad & z = f(\mathbf{x}) = 3x_1 + 4x_2 \\ \text{s.t.} \quad & 2x_1 + x_2 \leq 6 \\ & 2x_1 + 3x_2 \leq 9 \\ \text{with} \quad & x_1, x_2 \text{ nonnegative and integral.} \end{aligned}$$

Basis	x_1	x_2	s_1	s_2	s_3	s_4	s_5	rhs
Subproblem 1								
z	-3	-4	0	0				0
s_1	2	1	1	0				6
s_2	2	$\boxed{3}$	0	1				9
z	$-\frac{1}{3}$	0	0	$\frac{4}{3}$				12
s_1	$\frac{4}{3}$	0	1	$-\frac{1}{3}$				3
x_2	$\frac{2}{3}$	1	0	$\frac{1}{3}$				3
z	0	0	$\frac{1}{4}$	$\frac{5}{4}$				$12\frac{3}{4}$
x_1	1	0	$\frac{3}{4}$	$-\frac{1}{4}$				$\frac{9}{4}$
x_2	0	1	$-\frac{1}{2}$	$\frac{1}{2}$				$\frac{3}{2}$
Subproblem 2								
$x_2 \leq 1$								
z	0	0	$\frac{1}{4}$	$\frac{5}{4}$	0			$12\frac{3}{4}$
x_1	1	0	$\frac{3}{4}$	$-\frac{1}{4}$	0			$\frac{9}{4}$
x_2	0	1	$-\frac{1}{2}$	$\frac{1}{2}$	0			$\frac{3}{2}$
s_3	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	1			$-\frac{1}{2}$
z	0	0	$\frac{3}{2}$	0	$\frac{5}{2}$			$11\frac{1}{2}$
x_1	1	0	$\frac{1}{2}$	0	$-\frac{1}{2}$			$\frac{5}{2}$
x_2	0	1	0	0	1			1
s_2	0	0	-1	1	-2			1
Subproblem 4								
$x_1 \leq 2$								
z	0	0	$\frac{3}{2}$	0	$\frac{5}{2}$	0		$11\frac{1}{2}$
x_1	1	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	0		$\frac{10}{4}$
x_2	0	1	0	0	1	0		1
s_2	0	0	-1	1	-2	0		1
s_4	0	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	1		$-\frac{1}{2}$
z	0	0	0	0	4	3		10
x_1	1	0	0	0	0	1		2
x_2	0	1	0	0	1	0		1
s_2	0	0	0	1	-3	-2		2
s_1	0	0	1	0	-1	-2		1

With the solution of the subproblem 4, we have a candidate for the optimal solution, i.e. lower bound, $x_1^* = 2$, $x_2^* = 1$ and $LB = 10$.

Basis	x_1	x_2	s_1	s_2	s_3	s_4	s_5	rhs
Subproblem 5								
$x_1 \geq 3$								
z	0	0	$\frac{3}{2}$	0	$\frac{5}{2}$		0	$11\frac{1}{2}$
x_1	1	0	$\frac{1}{2}$	0	$-\frac{1}{2}$		0	$\frac{10}{4}$
x_2	0	1	0	0	1		0	1
s_2	0	0	-1	1	-2		0	1
s_5	0	0	$\frac{1}{2}$	0	$-\frac{1}{2}$		1	$-\frac{1}{2}$
z	0	0	4	0	0		5	9
x_1	1	0	0	0	0		-1	3
x_2	0	1	1	0	0		2	0
s_2	0	0	-3	1	0		-4	3
s_3	0	0	-1	0	1		-2	1

The feasible solution obtained from the subproblem 5 is not better that the current lower bound $LB = 10$, so subproblem 5 shall be fathomed.
All possibilities on the first side of the first branch have now been solved or fathomed. Then we return to the second side of the first branch.

Basis	x_1	x_2	s_1	s_2	s_6	s_7	s_8	rhs
z	0	0	$\frac{1}{4}$	$\frac{5}{4}$				$12\frac{3}{4}$
x_1	1	0	$\frac{3}{4}$	$-\frac{1}{4}$				$\frac{9}{4}$
x_2	0	1	$-\frac{1}{2}$	$\frac{1}{2}$				$\frac{3}{2}$
Subproblem 3								
$x_2 \geq 2$								
z	0	0	$\frac{1}{4}$	$\frac{5}{4}$	0			$12\frac{3}{4}$
x_1	1	0	$\frac{3}{4}$	$-\frac{1}{4}$	0			$\frac{9}{4}$
x_2	0	1	$-\frac{1}{2}$	$\frac{1}{2}$	0			$\frac{3}{2}$
s_6	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	1			$-\frac{1}{2}$
z	0	0	0	$\frac{3}{2}$	$\frac{1}{2}$			$12\frac{1}{2}$
x_1	1	0	0	$\frac{1}{2}$	$\frac{3}{2}$			$\frac{3}{2}$
x_2	0	1	0	0	-1			2
s_1	0	0	1	-1	-2			1
Subproblem 6								
$x_1 \leq 1$								
z	0	0	0	$\frac{3}{2}$	$\frac{1}{2}$	0		$12\frac{1}{2}$
x_1	1	0	0	$\frac{1}{2}$	$\frac{3}{2}$	0		$\frac{3}{2}$
x_2	0	1	0	0	-1	0		2
s_1	0	0	1	-1	-2	0		1
s_7	0	0	0	$-\frac{1}{2}$	$-\frac{3}{2}$	1		$-\frac{1}{2}$
z	0	0	0	$\frac{4}{3}$	0	$\frac{1}{3}$		$12\frac{1}{3}$
x_1	1	0	0	0	0	1		1
x_2	0	1	0	$\frac{1}{3}$	0	$-\frac{2}{3}$		$2\frac{1}{3}$
s_1	0	0	1	$-\frac{1}{3}$	0	$-\frac{4}{3}$		$1\frac{2}{3}$
s_6	0	0	0	$\frac{1}{3}$	1	$-\frac{2}{3}$		$\frac{1}{3}$

Hence, we branch on x_2 again, giving either $x_2 \leq 2$ (Subproblem 8) with $x_2^* = 2, x_1^* = 1, z^* = 11$ or $x_2 \geq 3$ (Subproblem 9) with $x_2^* = 3, x_1^* = 0, z^* = 12$. So our LB was updated from 10 to 11, and then from 11 to 12. Finally, we need to consider $x_1 \geq 2$ (Subproblem 7) from the second side, second branch.

Basis	x_1	x_2	s_1	s_2	s_6	s_7	s_8	rhs
Subproblem 7								
$x_1 \geq 2$								
z	0	0	0	$\frac{3}{2}$	$\frac{1}{2}$		0	$12\frac{1}{2}$
x_1	1	0	0	$\frac{1}{2}$	$\frac{3}{2}$		0	$\frac{3}{2}$
x_2	0	1	0	0	-1		0	2
s_1	0	0	1	-1	-2		0	1
s_8	0	0	0	$\frac{1}{2}$	$\frac{3}{2}$		1	$-\frac{1}{2}$

As there is no negative denominator in the ratio for the dual simplex method, this subproblem is infeasible and shall be fathomed. All the subproblems in our branching tree are solved or fathomed, so we can claim that our current LB=12, which comes from the candidate optimal solution $x_1^* = 0, x_2^* = 3$, $z^* = 12$, is the optimal solution.

2. Solve

$$\begin{array}{rl} \min & z = f(\mathbf{x}) = x_1 + x_2 \\ \text{s.t.} & 2x_1 + 2x_2 \geq 5 \\ & 12x_1 + 5x_2 \leq 30 \\ \text{with} & x_1, x_2 \text{ nonnegative and integral.} \end{array}$$

Basis	x_1	x_2	s_1	s_2	s_3	s_4	rhs
Subproblem 1							
z	-1	-1	0	0			0
s_1	-2	-2	1	0			-5
s_2	12	5	0	1			30
z	0	0	$-\frac{1}{2}$	0			$\frac{5}{2}$
x_1	1	1	$-\frac{1}{2}$	0			$\frac{5}{2}$
s_2	0	-7	6	1			0
Subproblem 2							
$x_1 \leq 2$							
z	0	0	$-\frac{1}{2}$	0	0		$\frac{5}{2}$
x_1	1	1	$-\frac{1}{2}$	0	0		$\frac{5}{2}$
s_2	0	-7	6	1	0		0
s_3	0	-1	$\frac{1}{2}$	0	1		$-\frac{1}{2}$
z	0	0	$-\frac{1}{2}$	0	0		$\frac{5}{2}$
x_1	1	0	0	0	1		2
s_2	0	0	$\frac{5}{2}$	1	-7		$\frac{7}{2}$
x_2	0	1	$-\frac{1}{2}$	0	-1		$\frac{1}{2}$
Subproblem 4							
$x_2 \geq 1$							
z	0	0	$-\frac{1}{2}$	0	0	0	$\frac{5}{2}$
x_1	1	0	0	0	1	0	2
s_2	0	0	$\frac{5}{2}$	1	-7	0	$\frac{7}{2}$
x_2	0	1	$-\frac{1}{2}$	0	-1	0	$\frac{1}{2}$
s_4	0	0	$-\frac{1}{2}$	0	-1	1	$-\frac{1}{2}$
z	0	0	0	0	1	-1	3
x_1	1	0	0	0	1	0	2
s_2	0	0	0	1	-12	5	1
x_2	0	1	0	0	0	-1	1
s_1	0	0	1	0	2	-2	1
z	0	0	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	$\frac{5}{2}$
x_1	1	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{3}{2}$
s_2	0	0	6	1	0	-1	7
x_2	0	1	0	0	0	-1	1
s_3	0	0	$\frac{1}{2}$	0	1	$-\frac{1}{2}$	$\frac{1}{2}$

Hence, we branch on x_1 again, giving either $x_1 \leq 1$ (Subproblem 6) with $x_1^* = 1, x_2^* = 2, z^* = 3$ or $x_1 \geq 2$ (Subproblem 7) with $x_1^* = 2, x_2^* = 1, z^* = 3$. So we have a candidate for the optimal solution with $x_1^* = 1, x_2^* = 2, z^* = 3$ and the current UB=3.

Then we turn to solve subproblem 5 by adding constraint $x_2 = 0$ to subproblem 2 as shown below. Notice that since this added constraint is an equality we don't need to introduce an extra slack or surplus variable.

Basis	x_1	x_2	s_1	s_2	s_3	rhs
z	0	0	$-\frac{1}{2}$	0	0	$\frac{5}{2}$
x_1	1	0	0	0	1	2
s_2	0	0	$\frac{5}{2}$	1	-7	$\frac{7}{2}$
x_2	0	1	$-\frac{1}{2}$	0	-1	$\frac{1}{2}$
Subproblem 5						
$x_2 = 0$						
z	0	0	$-\frac{1}{2}$	0	0	$\frac{5}{2}$
x_1	1	0	0	0	1	2
s_2	0	0	$\frac{5}{2}$	1	-7	$\frac{7}{2}$
x_2	0	1	$-\frac{1}{2}$	0	-1	$\frac{1}{2}$
s_3	0	0	$\frac{1}{2}$	0	1	$-\frac{1}{2}$

As there is no negative denominator in the ratio for the dual simplex method, this subproblem is infeasible and shall be fathomed.
Finally, we need to consider $x_1 \geq 3$ (Subproblem 3) from the second side, first branch.

Basis	x_1	x_2	s_1	s_2	s_5	rhs
Subproblem 3						
$x_1 \geq 3$						
z	0	0	$-\frac{1}{2}$	0	0	$\frac{5}{2}$
x_1	1	1	$-\frac{1}{2}$	0	0	$\frac{10}{4}$
s_2	0	-7	6	1	0	0
s_5	0	1	$-\frac{1}{2}$	0	1	$-\frac{1}{2}$
z	0	-1	0	0	-1	3
x_1	1	0	0	0	-1	3
s_2	0	5	0	1	12	-6
s_1	0	-2	1	0	-2	1

After completing one-step dual simplex procedure, we fixed one negative RHS but generate another negative RHS. While we tried to apply the dual simplex procedure again, we found that there exists no negative denominator in the ratio for the dual simplex method. So this subproblem is infeasible and shall be fathomed. All the subproblems in our branching tree are solved or fathomed, so we can claim that our current UB=3, which comes from the candidate optimal solution $x_1^* = 1, x_2^* = 2$, $z^* = 3$, is the optimal solution.

3. (Winston Sections 7, Review Problems, Group A. Problem 2 Page 407)

Five workers are available to perform four jobs. . The time it takes each worker to do each job is shown in table below

	Time (hours)			
Worker	Job 1	Job 2	Job 3	Job 4
1	10	15	10	15
2	12	8	20	16
3	12	9	12	18
4	6	12	15	18
5	16	12	8	12

Each worker is assigned no more than one job. The goal is to assign workers to jobs so as to minimise the total time required to perform the four jobs.

Use the Hungarian method to solve the problem.

Solution

After adding a dummy job column we obtain the following cost matrix:

						Row min
	10	15	10	15	0	0
	12	8	20	16	0	0
	12	9	12	18	0	0
	6	12	15	18	0	0
	16	12	8	12	0	0
Col. Min	6	8	8	12	0	

After subtracting the row minima (which are all 0) and then column minima we obtain the following reduced cost matrix:

4	7	2	3	0
6	0	12	4	0
6	1	4	6	0
0	4	7	6	0
10	4	0	0	0

All zeros in this matrix can be covered by just four lines in columns 1, 2, 5, and in row 5.

The smallest uncovered cost is 2. So we obtained

4	7	0	1	0
6	0	10	2	0
6	1	2	4	0
0	4	5	4	0
12	6	0	0	2

Five lines are needed to cover all zeros in this matrix, so an optimal assignment is available: worker 1 does job 3, worker 2 does job 2, worker 4 does job 1, worker 5 does job 4, and worker 3 does no job. A total of 36 hours required.

4. (Winston Sections 7, Review Problems, Group A. Problem 6 Page 408)

The Gotham City police have just received three calls for police. Five cars are available. The distance (in city blocks) of each car from each call is given in the table below.

	Distance (Blocks)		
Car	Call 1	Call 2	Call 3
1	10	11	18
2	6	7	7
3	7	8	5
4	5	6	4
5	9	4	7

Gotham City wants to minimise the total distance cars must travel to respond to the three police calls.

Use the Hungarian method to determine which car should respond to which call.

Solution

After adding two dummy "calls" column we obtain the following cost matrix:

						Row min
	10	11	18	0	0	0
	6	7	7	0	0	0
	7	8	5	0	0	0
	5	6	4	0	0	0
	9	4	7	0	0	0
Col. Min	5	4	4	0	0	

After subtracting the row minima (which are all 0) and then column minima we obtain the following reduced cost matrix:

5	7	14	0	0
1	3	3	0	0
2	4	1	0	0
0	2	0	0	0
4	0	3	0	0

All zeros in this matrix can be covered by just four lines in columns 4, 5, and in rows 4, 5.

The smallest uncovered cost is 1. So we obtained

4	6	13	0	0
0	2	2	0	0
1	3	0	0	0
0	2	0	1	1
4	0	3	1	1

Five lines are needed to cover all zeros in this matrix, so an optimal assignment is available. One optimal solution is: car 5 responding to call 2, car 4 responding to call 1, car 3 responding to call 3. Cars 1 and 2 do not respond to any calls. A total of 14 blocks will be traveled.

Note. An alternative solution is: car 2 responding to call 1, car 4 responding to call 3, car 5 responding to call 2. Cars 1 and 3 do not respond to any calls.