Tutorial 4

• RHS of constr are ≥ 0

• constraints are $=$

• variables ≥ 0

1. Consider the LP in standard form:

$$
\begin{array}{rrcrcrcrcrcrcl}
\max z = & x_1 & + & x_2 & + & x_3 & & & & & & & & \\
s.t. & x_1 & + & 2x_2 & + & 3x_3 & + & s_1 & & & & & = & 5 \\
 & 2x_1 & + & 4x_2 & + & x_3 & & & + & s_2 & & & = & 6 \\
 & x_1 & + & 3x_2 & + & 3x_3 & & & & & + & s_3 & = & 6 \\
 & \multicolumn{13}{l}{x_1, x_2, x_3, s_1, s_2, s_3 \geq 0.}
\end{array}
$$

   (a) If the basis $\mathbf{x}_B$ is (s_1, s_2, s_3), what are the values for $\mathbf{x}_B$ and $\mathbf{x}_N$, and what is the objective value?

   (b) If the basis $\mathbf{x}_B$ is (x_1, x_2, x_3), what are the values for $\mathbf{x}_B$ and $\mathbf{x}_N$, and what is the objective value?

   (c) If the basis $\mathbf{x}_B$ is (x_1, x_2, s_1), what are the values for $\mathbf{x}_B$ and $\mathbf{x}_N$. Is this a basic feasible solution (bfs)?

2. Solve the following LP with Simplex method in the tabular form:

$$
\begin{array}{rrcrl}
\min z = & -x_1 & - & 2x_2 & \\
s.t. & x_1 & + & x_2 & \leq 2 \\
 & 2x_1 & + & x_2 & \leq 1
\end{array}
$$

$$
\begin{array}{lrcrcr}
\max & 2x_1 & + & x_2 & & \\
\text{s.t.} & 7x_1 & + & 4x_2 & \leq & 14 \\
 & x_1 & & & \leq & 2 \\
 & \multicolumn{5}{l}{x_1, x_2 \geq 0.}
\end{array}
$$

3. Show that the set of all $\mathbf{x}$ satisfying

$$
\begin{aligned}
\mathbf{Ax} &\leq \mathrm{b} \\
\mathbf{x} &\geq \mathbf{0}
\end{aligned}
$$

is convex. (Hint: take two arbitrary points satisfying the constraints, and consider a convex combination of these points)

- RHS of constr are ≥ 0
- constraints are $=$

1. Consider the LP in standard form:

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\begin{array}{rrcrcrcrcrcrcl}
\max z = & x_1 & + & x_2 & + & x_3 & & & & & & & & \\
s.t. & x_1 & + & 2x_2 & + & 3x_3 & + & s_1 & & & & & = & 5 \\
& 2x_1 & + & 4x_2 & + & x_3 & & & + & s_2 & & & = & 6 \\
& x_1 & + & 3x_2 & + & 3x_3 & & & & & + & s_3 & = & 6 \\
\multicolumn{14}{l}{\quad x_1, x_2, x_3, s_1, s_2, s_3 \geq 0.}
\end{array}
$$

- variables ≥ 0

(a) If the basis $\mathbf{x}_B$ is (s_1, s_2, s_3), what are the values for $\mathbf{x}_B$ and $\mathbf{x}_N$, and what is the objective value?

(b) If the basis $\mathbf{x}_B$ is (x_1, x_2, x_3), what are the values for $\mathbf{x}_B$ and $\mathbf{x}_N$, and what is the objective value?

(c) If the basis $\mathbf{x}_B$ is (x_1, x_2, s_1), what are the values for $\mathbf{x}_B$ and $\mathbf{x}_N$. Is this a basic feasible solution (bfs)?

a) N of variables $-$ $n = 6$

N of constraints $-$ $m = 3$

$|x_B| = m = 3 \qquad |x_N| = n - m = 3$

$x_B = (s_1 \ s_2 \ s_3) \rightarrow x_N = (x_1 \ x_2 \ x_3)$

$x_1 = x_2 = x_3 = 0.$

$s_1 = 5$

$s_2 = 6$

$s_3 = 6$

$z(x_B) = 0 \qquad z(x_B) = c_B^T B^{-1} b$

$c_B^T = \underset{s_1 \ s_2 \ s_3}{(0, 0, 0)}$

$z = x_1 + x_2 + x_3 + \cancel{0} s_1 + \cancel{0} s_2 + \cancel{0} s_3$

$c^T = (\underbrace{1 \quad 1 \quad 1}_{c_N^T} \quad \underbrace{0 \quad 0 \quad 0}_{c_B^T})$

c) $x_B = (x_1 \ x_2 \ s_1) \qquad x_N = (x_3 \ s_2 \ s_3)$

$x_3 = s_2 = s_2 = 0$

$$B = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \\ 1 & 3 & 0 \end{pmatrix}$$

$x_B = B^{-1} b \rightarrow B^{-1} - ?$

$$\left(B \mid b\right) \overset{ERO}{\sim} \left(I \mid B^{-1}b\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 2 & 4 & 0 & 6 \\ 1 & 3 & 0 & 6 \end{array}\right) \underset{\substack{R_2' = R_2 - 2R_1 \\ R_3' = R_3 - R_1}}{\overset{R_1' = R_1}{\sim}} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 0 & 0 & -2 & -4 \\ 0 & 1 & -1 & 1 \end{array}\right)$$

$$\underset{R_2 \Leftrightarrow R_3}{\overset{\text{swap}}{\sim}} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -2 & -4 \end{array}\right) \underset{\substack{R_2' = R_2 + R_3' \\ R_3' = R_3 / (-2)}}{\overset{R_1' = R_1 - 2R_2' - R_3}{\sim}} \left(\begin{array}{ccc|l} 1 & 0 & 0 & 5-6-2 = -3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$$x_B = \begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix} < 0 \rightarrow \text{not feasible}$$

2. Solve the following LP with Simplex method in the tabular form:

$$
\begin{array}{rl}
\min z = & -x_1 - 2x_2 \\
s.t. & x_1 + x_2 \le 2 \\
& 2x_1 + x_2 \le 1 \\
& x_1, x_2 \ge 0.
\end{array}
$$

$$
\begin{array}{rl}
\max & 2x_1 + x_2 \\
\text{s.t.} & 7x_1 + 4x_2 \le 14 \\
& x_1 \le 2 \\
& x_1, x_2 \ge 0.
\end{array}
$$

$$
\begin{array}{rl}
\min z = & -x_1 - 2x_2 \\
s.t. & x_1 + x_2 \le 2 \\
& 2x_1 + x_2 \le 1
\end{array}
$$

$$c^T = (\underbrace{-1, -2}_{C_{N_0}^T}, \underbrace{0, 0}_{C_{B_0}^T})$$

1. Standard form:

$$
\begin{array}{rl}
\min & z = -x_1 - 2x_2 + 0S_1 + 0S_2 \\
s.t. & x_1 + x_2 + S_1 = 2 \\
& 2x_1 + x_2 + S_2 = 1 \\
& x_1\ x_2\ S_1\ S_2 \ge 0
\end{array}
$$

2. $n = 4$; $m = 2 \rightarrow |x_B| = 2$

$|x_N| = 2$

$x_{B_0} = (S_1\ S_2)$

	x_N	x_B	RHS
z	$C_B^T B^{-1} N - C_N^T$	0^T	$C_B^T B^{-1} b$
x_B	$B^{-1} N$	I	$B^{-1} b$

canonical form

$$B_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I = B^{-1}$$

$$\hat{C}_{N_0}^T = \underbrace{C_{B_0}^T}_{\substack{\| \\ 0}} B_0^{-1} N - C_{N_0}^T = -C_{N_0}^T = (1, 2)$$

$$z(x_{B_0}) = \underbrace{C_{B_0}^T}_{} B^{-1} b = 0$$

$$z = -x_1 - 2x_1$$

$$\downarrow$$

$$z + x_1 + 2x_1 = 0.$$

3. Initial Tableau:

	x_1	x_2	s_1	s_2	RHS
z	1	2	0	0	0
s_1	1	1	1	0	2
s_2	2	1	0	1	1
z	-3	0	0	-2	-2
s_1	-1	0	1	-1	1
x_2	2	1	0	1	1

Ratio Test

$2/1$

vs

$1/1 \rightarrow s_2$ is leaving

$R_0' = R_0 - 2R_2$

$R_1' = R_1 - R_2$

$R_2' = R_2$

The Tableau is optimal as all $c_N^T \leq 0$

$z^* = -2$

$x_1^* = 0$ as x_1 is non-basic

$x_2^* = 1$

$s_1^* = 1$

$s_2 = 0$ as s_2 is non-basic

3. Show that the set of all $\mathbf{x}$ satisfying

$$\mathbf{Ax} \leq \mathrm{b}$$
$$\mathbf{x} \geq \mathbf{0}$$

is convex. (Hint: take two arbitrary points satisfying the constraints, and consider a convex combination of these points)

Let $x^{(1)}$ and $x^{(2)}$ are :

$$Ax^{(1)} \leq b$$
$$Ax^{(2)} \leq b$$

$$\tilde{x} = \underbrace{\alpha x^{(1)} + (1-\alpha) x^{(2)}}_{\text{point on the Interval between } x^{(1)} \text{ and } x^{(2)}}, \quad 0 < \alpha < 1$$

$$A\tilde{x} = \alpha \underbrace{Ax^{(1)}}_{\leq b} + (1-\alpha) \underbrace{Ax^{(2)}}_{\leq b} \leq$$
$$\leq \alpha b + (1-\alpha) b = b$$

$\downarrow$

$$A\tilde{x} \leq b$$

$\downarrow$

$\tilde{x}$ is in the set $\rightarrow$ the set is convex

$\longrightarrow$ Show that set of vectors $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ satisfying $(*)$ is convex

$(*)$ $5x_1 + 3x_2 \leq 1$

1. Write $(*)$ in a vector form $ax \leq b$

$$(5, 3)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq 1$$

$A \quad x \quad b$

Let $x^{(1)}$ $x^{(2)}$:

$$(5,3)\begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \end{pmatrix} \le 1 \qquad x^{(1)} = \begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \end{pmatrix}$$

$$(5,3)\begin{pmatrix} x_1^{(2)} \\ x_2^{(2)} \end{pmatrix} \le 1 \qquad x^{(2)} = \begin{pmatrix} x_1^{(2)} \\ x_2^{(2)} \end{pmatrix}$$

$$\tilde{x} = \underbrace{\alpha x^{(1)} + (1-\alpha) x^{(2)}}_{\text{point on the Interval between } x^{(1)} \text{ and } x^{(2)}}, \quad 0 < \alpha < 1$$

$$(5,3)\tilde{x} = \alpha \underbrace{(5,3) x^{(1)}}_{\le 1} + (1-\alpha)\underbrace{(5,3) x^{(2)}}_{\le 1} \le$$

$$\le \alpha + (1-\alpha) = 1$$

$\downarrow$

$(5,3)\tilde{x} \le 1 \rightarrow \tilde{x}$ is in the set,

the set is convex