37242 Introduction to Optimisation

Tutorial 4

1. (a) From the Simplex algebraic formulae, we have:
   $\mathbf{x_B} = \mathbf{B^{-1}b}$, $\mathbf{x_N} = \mathbf{0}$, and $z = \mathbf{c_B^T B^{-1} b}$
   If the basis is $\mathbf{x_B} = (s_1, s_2, s_3)$, then $\mathbf{B} = \mathbf{I}$.

$$\mathbf{x_B} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 6 \end{bmatrix}, \mathbf{x_N} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

   The objective is 0.

   (b) If the basis is $\mathbf{x_B} = (x_1, x_2, x_3)$, then $\mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 1 & 3 & 3 \end{bmatrix}$.

$$\mathbf{B}^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 1 & 3 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{9}{5} & \frac{3}{5} & -2 \\ -1 & 0 & 1 \\ \frac{2}{5} & \frac{-1}{5} & 0 \end{bmatrix}.$$

   Now

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} \frac{9}{5} & \frac{3}{5} & -2 \\ -1 & 0 & 1 \\ \frac{2}{5} & \frac{-1}{5} & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 6 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ 1 \\ \frac{4}{5} \end{bmatrix},$$

   $\mathbf{x_N} = \mathbf{0}$, and the objective is

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T \begin{bmatrix} \frac{3}{5} \\ 1 \\ \frac{4}{5} \end{bmatrix} = \frac{12}{5}.$$

   (c) We demonstrate another solving approach. If the basis is (x_1, x_2, s_1), then the nonbasic variables are $x_3 = s_2 = s_3 = 0$. After the substitution, we have

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \\ 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ s_1 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 6 \end{bmatrix}$$

   Solving it by Gauss-Jordan elimination gives $x_1 = -3, x_2 = 3, s_1 = 2$. This basic solution is **not** feasible, since x_1 is required to be nonnegative.

2b. The first step is to convert the general form to standard form:

$$\begin{array}{rrcrcrcrcl} \max z = & 2x_1 & + & x_2 & & & & & & \\ \text{s.t.} & 7x_1 & + & 4x_2 & + & s_1 & & & = & 14 \\ & x_1 & & & & & + & s_2 & = & 2 \\ & \multicolumn{9}{l}{x_1, x_2, s_1, s_2 \geq 0.} \end{array}$$

Then you can directly perform the Simplex method in tabular form to solve it. But here I would like to demonstrate the simplex algebraic formulae to strengthen your impression.

An initial bfs has

$$\mathbf{x_B} = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 14 \\ 2 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{ and } \mathbf{c_B}^T = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

The reduced costs for the two current nonbasic variable x_1 and x_2 are:

$$\hat{c}_1 = \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 1 \end{bmatrix} - 2 = -2, \text{ and}$$

$$\hat{c}_2 = \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} - 1 = -1.$$

So the entering variable is x_1. Then to select the leaving variable, we take

$$\widehat{\mathbf{b}} = \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 14 \\ 2 \end{bmatrix} = \begin{bmatrix} 14 \\ 2 \end{bmatrix}, \text{ and}$$

$$\widehat{\mathbf{A}}_t = \mathbf{B}^{-1}\mathbf{A}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}.$$

Since we have a tie for the ratio test $\min\{\frac{14}{7}, \frac{2}{1}\}$, we can select either as the leaving variable, say s_2.

Now we go back to the Simplex procedure in tabular form.

basis	x_1	x_2	s_1	s_2	rhs	
z	-2	-1	0	0	0	
s_1	7	4	1	0	14	
s_2	$\boxed{1}$	0	0	1	2	
z	0	-1	0	2	4	$R_0' \leftarrow R_0 + 2 \times R_2$
s_1	0	$\boxed{4}$	1	-7	0	$R_1' \leftarrow R_1 - 7 \times R_2$
x_1	1	0	0	1	2	
z	0	0	$\frac{1}{4}$	$\frac{1}{4}$	4	$R_0'' \leftarrow R_0' + R_1''$
x_2	0	1	$\frac{1}{4}$	$-\frac{7}{4}$	0	$R_1'' \leftarrow \frac{1}{4} \times R_1'$
x_1	1	0	0	1	2	

Since no negative coefficient exists in row 0, this tableau is the final optimal tableau. Notice that after the second iteration the z-value didn't get improved. This is "degeneracy." However, we still obtain an optimal solution $x_1 = 2, x_2 = 0, s_1 = s_2 = 0$ with $z_{\max} = 4$.

3. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be arbitrary vectors in the set such that

$$\mathbf{A}\mathbf{x}_1 \leq \mathrm{b}$$
$$\mathbf{x}_1 \geq \mathbf{0}, \text{ and}$$

$$\mathbf{A}\mathbf{x}_2 \leq \mathrm{b}$$
$$\mathbf{x}_2 \geq \mathbf{0}.$$

Then for any $0 \leq \lambda \leq 1$,

$$\lambda\mathbf{A}\mathbf{x}_1 + (1-\lambda)\mathbf{A}\mathbf{x}_2 \leq \lambda\mathbf{b} + (1-\lambda)\mathbf{b} = \mathbf{b}, \text{ and}$$
$$\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2 \geq \mathbf{0}.$$

So any convex combination of any two elements of the set is in the set. (For some sets in Euclidean two-dimensional space, the convex combination of any two points in that set is any point between the line segment connected by those two points.) Hence the set is convex.