37242 Introduction to Optimisation

Tutorial 5

1. Consider the LP

$$\begin{array}{lrcrcr} \min & -x_1 & - & 2x_2 & & \\ \text{s.t.} & x_1 & + & x_2 & \geq & 4 \\ & 3x_1 & + & 2x_2 & \leq & 24 \\ & \multicolumn{5}{l}{x_1, x_2 \geq 0.} \end{array}$$

   Solve this LP with two phase Simplex method (and compare the solution with that of the big-M method).

2. Use the two phase Simplex method or big-M method to determine whether there is a feasible solution to the following LP (Note: You do **not** need to find an optimal solution).

$$\begin{array}{lrcrcrcr} \min & 3x_1 & - & 2x_2 & & & & \\ s.t. & x_1 & + & 4x_2 & + & 4x_3 & \leq & 2 \\ & & & x_2 & + & x_3 & = & 1 \\ & \multicolumn{7}{l}{x_1, x_2, x_3 \geq 0.} \end{array}$$

3. Solve the following LP with the two phase Simplex method (and compare the solution with that of the big-M method):

$$\begin{array}{lrcrcrcr} \min & 3x_1 & - & 2x_2 & + & x_3 & & \\ s.t. & -3x_1 & + & 3x_2 & + & 2x_3 & \geq & 10 \\ & 2x_1 & - & 2x_2 & - & x_3 & \geq & 16 \\ & \multicolumn{7}{l}{x_1, x_2, x_3 \geq 0.} \end{array}$$

- Asymmetric dual:

apply the following rules:

primal/dual constraint		dual/primal variable
consistent with normal form	$\Longleftrightarrow$	variable ≥ 0
reversed with normal form	$\Longleftrightarrow$	variable ≤ 0
equality constraint	$\Longleftrightarrow$	variable urs

Example 4*

- **Primal LP:**

$\max z = 6x_1 + x_2 + x_3$ not in the normal form

s.t.

$4x_1 + 3x_2 - 2x_3 = 1$ → as the constraint is „=" → y_1 - URS

$6x_1 - 2x_2 + 9x_3 \geq 9$ → as the constraint is in „reverse" form, $y_2 \leq 0$

$2x_1 + 3x_2 + 8x_3 \leq 5$ → as the constraint is consistent with normal form, $y_3 \geq 0$

$x_1 \geq 0,\ x_2 \leq 0, x_3 - urs$

- Dual LP:

1. max normal form:

max $c^T x$

s.t. $Ax \leq b$

$x \geq 0$

$x_1 \geq 0,\ x_2 \leq 0, x_3 - urs$

1st dual constraint is consistent with normal

2nd dual constraint is in „reverse" form

3rd dual constraint is in „=" form

$$c = \begin{pmatrix} 6 \\ 1 \\ 1 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 9 \\ 5 \end{pmatrix} \quad A = \begin{pmatrix} 4 & 3 & -2 \\ 6 & -2 & 9 \\ 2 & 3 & 8 \end{pmatrix}$$

Dual:

$$\min w = b^T y = y_1 + 9y_2 + 5y_3$$

s.t.

$$4y_1 + 6y_2 + 2y_3 \geq 6$$

$$3y_1 - 2y_2 + 3y_3 \leq 1$$

$$-2y_1 + 9y_2 + 8y_3 = 1$$

$$y_1 - \text{URS},\ y_2 \leq 0,\ y_3 \geq 0$$

1. Consider the LP

$$\begin{array}{rrcrcl} \min & -x_1 & - & 2x_2 & & \\ \text{s.t.} & x_1 & + & x_2 & \geq & 4 \\ & 3x_1 & + & 2x_2 & \leq & 24 \\ & x_1, x_2 \geq 0. & & & & \end{array}$$

Solve this LP with $\boxed{\text{two phase Simplex method}}$ (and compare the solution with that of the big-M method).

1. Standard form :

$$\begin{aligned} \min \quad & z = -x_1 - 2x_2 \\ \text{s.t.} \quad & x_1 + x_2 - e_1 + a_1 = 4 \\ & 3x_1 + 2x_2 + s_2 = 24 \\ & x_1, x_2, e_1, a_1, s_2 \geq 0. \end{aligned}$$

2. Phase I :

$$\begin{aligned} \min \quad & w = a_1 \\ \text{s.t.} \quad & x_1 + x_2 - e_1 + a_1 = 4 \qquad R_1 \\ & 3x_1 + 2x_2 + s_2 = 24 \qquad R_2 \\ & x_1, x_2, e_1, a_1, s_2 \geq 0. \end{aligned}$$

$x_{B_0} = (a_1, s_2)$

$w = a_1 \rightarrow w - a_1 = 0. \rightarrow$ not canonical form as a_1 is in the basis

$$\begin{array}{l} w - a_1 = 0 \\ + \; R_1 \\ \hline w + x_1 + x_2 - e_1 = 4 \end{array}$$

	x_1	x_2	e_1	s_2	a_1	RHS
w	1	1	-1	0	0	4
a_1	1	1	-1	0	1	4
s_2	3	2	0	1	0	24

leaving

Ratio: $4/1$ vs $24/3$

$R_0' = R_0 - R_1$

$R_1' = R_1$

$R_2' = R_2 - 3R_1$

	x_1	x_2	e_1	s_2	a_1	RHS
w	0	0	0	0	-1	0
x_1	1	1	-1	0	1	4
s_2	0	-1	3	1	-3	12

optimal, as $\hat{c}_N^T \leq 0$

$w^* = 0 \rightarrow$ proceed to phase II

Phase 2:

$z = -x_1 - 2x_2 \rightarrow z + x_1 + 2x_2 = 0$.

IN the basis

not in canonical form

	x_1	x_2	e_1	s_2	a_1		
z	1	2	0	0		0	$R_0' = R_0 - R_1$
x_1	1	1	-1	0	1	4	
s_2	0	-1	3	1	-3	12	
z	0	1	1	0		-4	
x_1	1	1	-1	0		4	
s_2	0	$-1 < 0$	3	1		12	
z	-1	0	2	0		-8	$R_0' = R_0 - R_1$
x_2	1	1	$-1 < 0$	0		4	$R_1' = R_1$
s_2	1	0	2	1		16	$R_2' = R_2 + R_1$
z	-2	0	0	-1		-24	$R_0' = R_0 - R_2$
x_2	$3/2$	1	0	$1/2$		12	$R_1' = R_1 + R_2'$
e_1	$1/2$	0	1	$1/2$		8	$R_2' = \frac{R_2}{2}$

optimal as $\hat{c}_N < 0$

$z^* = -24$

$x_1^* = 0$

$x_2^* = 12$

2. Use the two phase Simplex method or big-M method to determine whether there is a feasible solution to the following LP (Note: You do **not** need to find an optimal solution).

$$\begin{array}{llllllll} \min & 3x_1 & - & 2x_2 & & & & \\ s.t. & x_1 & + & 4x_2 & + & 4x_3 & \leq & 2 \\ & & & x_2 & + & x_3 & = & 1 \\ & x_1, x_2, x_3 \geq 0. & & & & & & \end{array}$$

1. St. Form:

$$\min \ z = 3x_1 - 2x_2$$

$$\begin{array}{llr} \text{s.t.} & x_1 + 4x_2 + 4x_3 + s_1 & = 2 \\ & x_2 + x_3 + a_2 & = 1 \\ & x_1 \ x_2 \ x_3 \ s_1 \ a_2 \geq 0 & \end{array}$$

2. Phase I: solve

$$\min \ w = a_2$$

$$\begin{array}{llrl} \text{s.t.} & x_1 + 4x_2 + 4x_3 + s_1 & = 2 & R_1 \\ & x_2 + x_3 + a_2 & = 1 & R_2 \\ & x_1 \ x_2 \ x_3 \ s_1 \ a_2 \geq 0 & & \end{array}$$

$$x_{B_0} = (s_1 \ a_2)$$

$w - a_2 = 0 \rightarrow$ not canonical

$$\begin{array}{l} \quad w - a_2 = 0 \\ + R_2 \\ \hline w + x_2 + x_3 = 1 \end{array}$$

	x_1	x_2	x_3	s_1	a_2	RHS	
w	0	1	1	0	0	1	
s_1	1	4	4	1	0	2	$2/4$ leaving
a_1	0	1	1	0	1	1	$1/1$
w	$-1/4$	0	0	$-1/4$	0	$1/2$	$R_0' = R_0 - R_1'$
x_2	$1/4$	1	1	$1/4$	0	$1/2$	$R_1' = \frac{R_1}{4}$
a_1	$-1/4$	0	0	$-1/4$	1	$1/2$	$R_2' = R_2 - R_1'$

optimal as $\hat{c}_N < 0$,

BUT $w^* = 1/2 \neq 0$ $\rightarrow$ STOP,

and $a_1 = 1/2 \neq 0$ no feas. sol-ns

2. Use the two phase Simplex method or big-M method to determine whether there is a feasible solution to the following LP (Note: You do **not** need to find an optimal solution).

$$\begin{array}{lrcrcrcl} \min & 3x_1 & - & 2x_2 & & & & \\ s.t. & x_1 & + & 4x_2 & + & 4x_3 & \leq & 2 \\ & & & x_2 & + & x_3 & = & 1 \\ & x_1, x_2, x_3 \geq 0. \end{array}$$

1. Standard form for Big M.

$$\min z = 3x_1 - 2x_2 + Ma_2$$

$$\text{s.t.} \quad x_1 + 4x_2 + 4x_3 + s_1 = 2 \quad R_1$$

$$x_2 + x_3 + a_2 = 1 \quad R_2$$

$$x_1\ x_2\ x_3\ s_1\ a_2 \geq 0.$$

$$x_{B_0} = (s_1\ a_2)$$

$$z - 3x_1 + 2x_2 - Ma_2 = 0 \rightarrow \text{not canonical}$$

$$+ MR_2$$

$$z - 3x_1 + 2x_2 \cancel{- Ma_2} + Mx_2 + Mx_3 + \cancel{Ma_2} = M$$

$$z - 3x_1 + (M+2)x_2 + Mx_3 = M$$

	x_1	x_2	x_3	s_1	a_2	RHS	
z	-3	$M+2$	M	0	0	M	
s_1	1	4	4	1	0	2	$2 \div 4$
a_2	0	1	1	0	1	1	$1 \div 1$
z	$-3-\frac{M+2}{4}$	0	-2	$-\frac{M+2}{4}$	0	$M-\left(\frac{M+2}{2}\right)$	$R_0' = R_0 - (M+2)$
x_2	$1/4$	1	1	$1/4$	0	$1/2$	$R_1' = R_1/4$

$$a_2 \left| \begin{matrix} -1/4 & 0 & 0 & -1/4 & 1 \end{matrix} \right| \; 1/2 \qquad R_2' = R_2 - R_1'$$

optimal Tableau : $c_N^T \leq 0$

$z^* = \frac{M}{2} - 1 \rightarrow$ not bounded

$$\left.\begin{aligned} x_1 &= 0 \\ x_2 &= 1/2 \\ x_3 &= 0 \end{aligned}\right\}$$ not feas. for original problem

$x_1 + x_2 + x_3 = 1/2 \neq 1$

$a_1 = 1/2 \neq 0.$

The problem is infeasible