37242 Introduction to Optimisation

Tutorial 6

1. We first put the problem into standard form, giving:

$$\begin{array}{rrcrcrcrcr} \min\ z = & -x_1 & - & 2x_2 & & & & & & \\ \text{s.t.} & x_1 & + & x_2 & - & s_1 & & & = & 4 \\ & 3x_1 & + & 2x_2 & & & + & s_2 & = & 24 \\ & \multicolumn{9}{l}{x_1, x_2, s_1, s_2 \geq 0.} \end{array}$$

Since s_2 can be used in an initial basis, we just need an artificial variable in the first row. So the Phase-I problem is

$$\begin{array}{rrcrcrcrcrcr} \min\ w = & & & & & & & & & a_1 & & \\ \text{s.t.} & x_1 & + & x_2 & - & s_1 & & & + & a_1 & = & 4 \\ & 3x_1 & + & 2x_2 & & & + & s_2 & & & = & 24 \\ & \multicolumn{11}{l}{x_1, x_2, s_1, s_2, a_1 \geq 0,} \end{array}$$

giving the tableau

	x_1	x_2	s_1	s_2	a_1	rhs
w	0	0	0	0	-1	0
a_1	1	1	-1	0	1	4
s_2	3	2	0	1	0	24

First we need to eliminate the nonzero row-0 coefficient in the column of the basic variable a_1 by some EROs. Then the initial Phase-I tableau is yielded.

	x_1	x_2	s_1	s_2	a_1	rhs	
w	1	1	-1	0	0	4	$R_0' \leftarrow R_0 + R_1$
a_1	$\boxed{1}$	1	-1	0	1	4	
s_2	3	2	0	1	0	24	

Do the pivoting by entering x_1 and leaving a_1.

	x_1	x_2	s_1	s_2	a_1	rhs	
w	0	0	0	0	-1	0	$R_0'' \leftarrow R_0' - R_1'$
x_1	1	1	-1	0	1	4	
s_2	0	-1	3	1	-3	12	$R_2'' \leftarrow R_2' - 3R_1'$

No positive reduced cost exists so this is the final Phase-I tableau. Since we have the optimal value $w_{\min} = 0$, there exists a bfs for the original LP. There is no artificial variables in the basis, so we can proceed to Phase II by dropping the column corresponding to a_1 and replacing the row 0 with the objective function of the original problem.

	x_1	x_2	s_1	s_2	rhs
z	1	2	0	0	0
x_1	1	1	-1	0	4
s_2	0	-1	3	1	12

Again, there is a nonzero row-0 coefficient in thee column of the basic variable x_1, so we need to cancel this out. The tableaux of subsequent Simplex iterations are shown as below.

	x_1	x_2	s_1	s_2	rhs	
z	0	1	1	0	-4	$R_0' \leftarrow R_0 - R_1$
x_1	1	$\boxed{1}$	-1	0	4	
s_2	0	-1	3	1	12	
z	-1	0	2	0	-8	$R_0'' \leftarrow R_0' - R_1'$
x_2	1	1	-1	0	4	
s_2	1	0	$\boxed{2}$	1	16	$R_2'' \leftarrow R_2' + R_1'$
z	-2	0	0	-1	-24	$R_0''' \leftarrow R_0'' - R_2''$
x_2	$\frac{3}{2}$	1	0	$\frac{1}{2}$	12	$R_1''' \leftarrow R_1'' + R_2'''$
s_1	$\frac{1}{2}$	0	1	$\frac{1}{2}$	8	$R_2''' \leftarrow \frac{1}{2}R_2''$

Hence, the optimal solution is $z = -24$ when $x_1 = 0, x_2 = 12, s_1 = 8$ and $s_2 = 0$.

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To solve it with the Big-M method, we first generate the following modified LP model

$$\begin{array}{rrrrrrrrrrr} \min\ z' = - & x_1 & - & 2x_2 & & & & + & Ma_1 & & \\ \text{s.t.} & x_1 & + & x_2 & - & s_1 & & + & a_1 & = & 4 \\ & 3x_1 & + & 2x_2 & & & + \ s_2 & & & = & 24 \\ & \multicolumn{10}{l}{x_1, x_2, s_1, s_2, a_1 \geq 0,} \end{array}$$

and then have the following initial simplex tableau satisfying the canonical form

	x_1	x_2	s_1	s_2	a_1	rhs
z'	$M+1$	$M+2$	$-M$	0	0	$4M$
a_1	1	$\boxed{1}$	-1	0	1	4
s_2	3	2	0	1	0	24
z'	-1	0	2	0		-8
x_2	1	1	-1	0		4
s_2	1	0	$\boxed{2}$	1		16
z'	-2	0	0	-1		-24
x_2	$\frac{3}{2}$	1	0	$\frac{1}{2}$		12
s_1	$\frac{1}{2}$	0	1	$\frac{1}{2}$		8

An identical solution to that using Two-phase is obtained.

2. We add a slack variable to the first row, which can serve as an initial basis element. Then we need a second basis element, so we add an artificial variable to the second row. The Phase I objective is then to minimize the value of this artificial variable.

$$\begin{array}{rrrrrrrrrrrr} \min\ w = & & & & & & & & & a_1 & & \\ s.t. & x_1 & + & 4x_2 & + & 4x_3 & + & s_1 & & & = & 2 \\ & & & x_2 & + & x_3 & & & + & a_1 & = & 1 \\ & \multicolumn{11}{l}{x_1, x_2, x_3, s_1, a_1 \geq 0.} \end{array}$$

(Note: you could add an artificial variable to the first row as well as the slack variable — this would still be correct, but would involve more work than absolutely necessary. However, if you add an artificial variable *instead of* the slack variable, then that is wrong!)

	x_1	x_2	x_3	s_1	a_1	rhs
w	0	0	0	0	-1	0
s_1	1	4	4	1	0	2
a_1	0	1	1	0	1	1

We need a zero row-0 coefficient in a_1's column to satisfy the canonical form and begin on the Simplex method:

	x_1	x_2	x_3	s_1	a_1	rhs	
w	0	1	1	0	0	1	$R_0' \leftarrow R_0 + R_2$
s_1	1	$\boxed{4}$	4	1	0	2	
a_1	0	1	1	0	1	1	
w	$-\frac{1}{4}$	0	0	$-\frac{1}{4}$	0	$\frac{1}{2}$	$R_0'' \leftarrow R_0' - R_1''$
x_2	$\frac{1}{4}$	1	1	$\frac{1}{4}$	0	$\frac{1}{2}$	$R_1'' \leftarrow \frac{1}{4}R_1'$
a_1	$-\frac{1}{4}$	0	0	$-\frac{1}{4}$	1	$\frac{1}{2}$	$R_2'' \leftarrow R_2' - R_1''$

Now, we have no negative reduced costs, so this is the optimal solution to the Phase-I LP. However, a_1 is still in the basis, and the optimal objective value is $w_{\min} = \frac{1}{2}$. This means that there is no feasible solution to the original problem.

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To solve it with the Big-M method, we first generate the following modified LP model

$$\begin{array}{rrcrcrcrcrcl}
\min\ z' = & 3x_1 & - & 2x_2 & & & & & + & Ma_1 & & \\
s.t. & x_1 & + & 4x_2 & + & 4x_3 & + & s_1 & & & = & 2 \\
 & & & x_2 & + & x_3 & & & + & a_1 & = & 1 \\
 & \multicolumn{11}{l}{x_1, x_2, x_3, s_1, a_1 \geq 0.}
\end{array}$$

and then have the following initial simplex tableau satisfying the canonical form

	x_1	x_2	x_3	s_1	a_1	rhs
z'	-3	$M+2$	M	0	0	M
s_1	1	$\boxed{4}$	4	1	0	2
a_1	0	1	1	0	1	1
z'	$-\frac{(M+14)}{4}$	0	-2	$-\frac{(M+2)}{4}$	0	$\frac{(M-2)}{2}$
x_2	$\frac{1}{4}$	1	1	$\frac{1}{4}$	0	$\frac{1}{2}$
a_1	$-\frac{1}{4}$	0	0	$-\frac{1}{4}$	1	$\frac{1}{2}$

There is no positive reduced cost on row 0 for further Simplex iteration, but the artificial variable a_1 is positive. So the original LP is infeasible.

3. We add excess(or surplus) variables, and then need to add an artificial variable to each row, leading to the Phase-I problem:

$$
\begin{array}{rrcrcrcrcrcrcrcl}
\min\ w = & & & & & & & & & & + & a_1 & + & a_2 & & \\
s.t. & -3x_1 & + & 3x_2 & + & 2x_3 & - & s_1 & & & + & a_1 & & & = & 10 \\
 & 2x_1 & - & 2x_2 & - & x_3 & & & - & s_2 & & & + & a_2 & = & 16 \\
 & \multicolumn{15}{l}{x_1, x_2, x_3, s_1, s_2, a_1, a_2 \geq 0.}
\end{array}
$$

So we put this into tableau form, cancel out the nonzero row-0 coefficients in the basic variables' columns, and then solve with the Simplex method:

	x_1	x_2	x_3	s_1	s_2	a_1	a_2	rhs	
w	0	0	0	0	0	-1	-1	0	
a_1	-3	3	2	-1	0	1	0	10	
a_2	2	-2	-1	0	-1	0	1	16	
w	-1	1	1	-1	-1	0	0	26	$R_0' \leftarrow R_0 + R_1 + R_2$
a_1	-3	3	$\boxed{2}$	-1	0	1	0	10	
a_2	2	-2	-1	0	-1	0	1	16	
w	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	0	21	$R_0'' \leftarrow R_0' - R_1''$
x_3	$-\frac{3}{2}$	$\frac{3}{2}$	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	5	$R_1'' \leftarrow \frac{1}{2}R_1'$
a_2	$\boxed{\frac{1}{2}}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	-1	$\frac{1}{2}$	1	21	$R_2'' \leftarrow R_2' + R_1''$
w	0	0	0	0	0	-1	-1	0	$R_0''' \leftarrow R_0'' - R_2''$
x_3	0	0	1	-2	-3	2	3	68	$R_1''' \leftarrow R_1'' + 3R_2''$
x_1	1	-1	0	-1	-2	1	2	42	$R_2''' \leftarrow 2R_2''$

Thus, we have reached the end of Phase I with an optimal solution $w_{\min} = 0$. No artificial variables remains in the optimal basis, so we dump the artificial variables and go back to the original objective function:

	x_1	x_2	x_3	s_1	s_2	rhs
z	-3	2	-1	0	0	0
x_3	0	0	1	-2	-3	68
x_1	1	-1	0	-1	-2	42

Again, we need a canonical form as below:

	x_1	x_2	x_3	s_1	s_2	rhs	
z	0	-1	0	-5	-9	194	$R_0' \leftarrow R_0 + R_1 + 3R_2$
x_3	0	0	1	-2	-3	68	
x_1	1	-1	0	-1	-2	42	

No Simplex iteration is needed since all the reduced costs are non-positive. This is optimal Phase-II tableau, and the optimal solution to the original problem is $z_{\min} = 194$ at $x_1 = 42, x_2 = 0$ and $x_3 = 68$.

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To solve it with the Big-M method, we first generate the following modified LP model

$$
\begin{array}{rrcrcrcrcrcrcrcl}
\min\ z' = & 3x_1 & - & 2x_2 & + & x_3 & & & & & & Ma_1 & + & Ma_2 & & \\
s.t. & -3x_1 & + & 3x_2 & + & 2x_3 & - & s_1 & & & + & a_1 & & & = & 10 \\
 & 2x_1 & - & 2x_2 & - & x_3 & & & - & s_2 & & & + & a_2 & = & 16 \\
 & \multicolumn{15}{l}{x_1, x_2, x_3, s_1, s_2, a_1, a_2 \geq 0.}
\end{array}
$$

and then have the following initial simplex tableau satisfying the canonical form

	x_1	x_2	x_3	s_1	s_2	a_1	a_2	rhs
z'	$-M-3$	$M+2$	$M-1$	$-M$	$-M$	0	0	$26M$
a_1	-3	$\boxed{3}$	2	-1	0	1	0	10
a_2	2	-2	-1	0	-1	0	1	16
z'	-1	0	$\frac{M-7}{3}$	$\frac{-2M+2}{3}$	$-M$		0	$\frac{68M-20}{3}$
x_2	-1	1	$\boxed{\frac{2}{3}}$	$-\frac{1}{3}$	0		0	$\frac{10}{3}$
a_2	0	0	$\frac{1}{3}$	$-\frac{2}{3}$	-1		1	$\frac{68}{3}$
z'	$\frac{M-9}{2}$	$\frac{-M+7}{2}$	0	$\frac{-M-1}{2}$	$-M$		0	$21M+5$
x_3	$-\frac{3}{2}$	$\frac{3}{2}$	1	$-\frac{1}{2}$	0		0	5
a_2	$\boxed{\frac{1}{2}}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	-1		1	21
z'	0	-1	0	-5	-9			194
x_3	0	0	1	-2	-3			68
x_1	1	-1	0	-1	-2			42

No Simplex iteration is needed since all the reduced costs are non-positive. This is optimal tableau, and the optimal solution to the original problem is $z_{\min} = 194$ at $x_1 = 42, x_2 = 0$ and $x_3 = 68$.