Tutorial 7

1. If we solve the LP

$$\begin{array}{rrcrcrcr} \min z = & -8x_1 & - & 3x_2 & + & x_3 & & \\ \text{s.t.} & x_1 & + & 2x_2 & - & x_3 & \leq & 4 \\ & 2x_1 & + & 5x_2 & & & \leq & 16 \\ & 4x_1 & & & + & x_3 & \leq & 12 \\ & \multicolumn{7}{l}{x_1, x_2, x_3 \geq 0,} \end{array}$$

then the Simplex method ends with the optimal solution, and the final tableau (adding slack variables s_1, s_2, s_3 to the three constraints) is:

basic	x_1	x_2	x_3	s_1	s_2	s_3	rhs
z	0	0	$-\frac{9}{8}$	$-\frac{3}{2}$	0	$-\frac{13}{8}$	$-25\frac{1}{2}$
x_2	0	1	$-\frac{5}{8}$	$\frac{1}{2}$	0	$-\frac{1}{8}$	$\frac{1}{2}$
s_2	0	0	$\frac{21}{8}$	$-\frac{5}{2}$	1	$\frac{1}{8}$	$\frac{15}{2}$
x_1	1	0	$\frac{1}{4}$	0	0	$\frac{1}{4}$	3

(a) By how much can the right-hand side of the first constraint increase and decrease without changing the optimal basis?

(b) By how much can the objective coefficient of x_3 increase and decrease without changing the optimal basis?

(c) By how much can the objective coefficient of x_1 increase and decrease without changing the optimal basis?

(d) By how much can the coefficient of x_3 in the first constraint increase and decrease without changing the optimal basis?

2. Consider the LP:

$$\begin{array}{rrrrrrrrrrr}
\min z = & -3x_1 & + & x_2 & - & 7x_3 & & & & & \\
\text{s.t.} & x_1 & + & 4x_2 & + & 2x_3 & + & x_4 & & = & 15 \\
 & 3x_1 & - & 2x_2 & + & 4x_3 & & & + \ x_5 & = & 5 \\
 & \multicolumn{10}{l}{x_1, x_2, x_3, x_4, x_5 \geq 0.}
\end{array}$$

The Simplex method ends with the optimal solution, and the final tableau is:

basic	x_1	x_2	x_3	x_4	x_5	rhs
z	-2	0	0	-0.5	-1.5	-15
x_2	-0.1	1	0	0.2	-0.1	2.5
x_3	0.7	0	1	0.1	0.2	2.5

(a) By how much can the objective coefficient of x_3 increase and decrease without changing the optimal basis?

(b) By how much can the coefficient of x_1 in the first constraint increase and decrease without changing the optimal basis?

(c) Suppose the right-hand side of the original problem is replaced by $\mathbf{b}' = \begin{bmatrix} 15 \\ 5+\Delta \end{bmatrix}$.

   i. What values can Δ take on without changing the optimal basis?

   ii. What is the optimal objective value (as a function of Δ) over the range found in part i. ?

(d) Solve the following LP:

$$\begin{array}{rrrrrrrrrrr}
\min & -3x_1 & + & x_2 & - & 7x_3 & & & & & \\
\text{s.t.} & x_1 & + & 4x_2 & + & 2x_3 & + & x_4 & & = & 16 \\
 & 3x_1 & - & 2x_2 & + & 4x_3 & & & + \ x_5 & = & 5 \\
 & \multicolumn{10}{l}{x_1, x_2, x_3, x_4, x_5 \geq 0.}
\end{array}$$

(e) Solve the following LP:

$$\begin{array}{rrrrrrrrrrr}
\min & -3x_1 & + & x_2 & - & 7x_3 & & & & & \\
\text{s.t.} & x_1 & + & 4x_2 & + & 2x_3 & + & x_4 & & = & 15 \\
 & 3x_1 & - & 2x_2 & + & 4x_3 & & & + \ x_5 & = & 6 \\
 & \multicolumn{10}{l}{x_1, x_2, x_3, x_4, x_5 \geq 0.}
\end{array}$$

(f) Compare the objective value in parts d) and e) to those in the initial problem, and compare the changes to the components in the vector $\mathbf{c_B}^T\mathbf{B}^{-1}$.

1. If we solve the LP

$$A = \begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 \\ \end{array}$$
$$A = \begin{pmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 2 & 5 & 0 & 0 & 1 & 0 \\ 4 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{rrrrrrrrr} \min z = & -8x_1 & - & 3x_2 & + & x_3 & & \\ \text{s.t.} & x_1 & + & 2x_2 & - & x_3 & \leq & 4 \\ & 2x_1 & + & 5x_2 & & & \leq & 16 \\ & 4x_1 & & & + & x_3 & \leq & 12 \\ & x_1, x_2, x_3 \geq 0, & & & & & & \end{array}$$

then the Simplex method ends with the optimal solution, and the final tableau (adding slack variables s_1, s_2, s_3 to the three constraints) is:

	B	B	N	N	B	N	
basic	x_1	x_2	x_3	s_1 B_0	s_2 B_0	s_3 B_0	rhs
z	0	0	$-\frac{9}{8}$	$-\frac{3}{2}$	0	$-\frac{13}{8}$	$-25\frac{1}{2}$
x_2	0	1	$-\frac{5}{8}$	$\frac{1}{2}$	0	$-\frac{1}{8}$	$\frac{1}{2}$
s_2	0	0	$\frac{21}{8}$	$-\frac{5}{2}$	1	$\frac{1}{8}$	$\frac{15}{2}$
x_1	1	0	$\frac{1}{4}$	0	0	$\frac{1}{4}$	3

$$X_B = (x_2 \; s_2 \; x_1)$$

$$B^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{8} \\ -\frac{5}{2} & 1 & \frac{1}{8} \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$$

$$X_{B_0} = (s_1 \; s_2 \; s_3)$$

$$B^{-1}N = \begin{pmatrix} -5/8 & 1/2 & -1/8 \\ 21/8 & -5/2 & 1/8 \\ 1/4 & 0 & 1/4 \end{pmatrix}$$

$$c = \begin{pmatrix} -8 \\ -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{array}{l} x_1 \\ x_2 \\ x_3 \\ s_1 \\ s_2 \\ s_3 \end{array}$$

$$N = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$
$$\begin{array}{ccc} x_3 & s_1 & s_3 \end{array}$$

$$b = \begin{pmatrix} 4 \\ 16 \\ 12 \end{pmatrix}$$

$$B^{-1}b = \begin{pmatrix} 1/2 \\ 15/2 \\ 3 \end{pmatrix} = x_B = \begin{pmatrix} x_2 \\ s_2 \\ x_1 \end{pmatrix} \begin{array}{c} = \\ = \\ = \end{array} \begin{pmatrix} 1/2 \\ 15/2 \\ 3 \end{pmatrix}$$

$$c_B^T B^{-1} = \begin{pmatrix} -\frac{3}{2} & 0 & -\frac{13}{8} \end{pmatrix}$$

(a) By how much can the right-hand side of the first constraint increase and decrease without changing the optimal basis?

(b) By how much can the objective coefficient of x_3 increase and decrease without changing the optimal basis?

(c) By how much can the objective coefficient of x_1 increase and decrease without changing the optimal basis?

(d) By how much can the coefficient of x_3 in the first constraint increase and decrease without changing the optimal basis?

a) $$b' = b + \begin{pmatrix} \Delta \\ 0 \\ 0 \end{pmatrix} \leftarrow \text{change in the } 1^{st} \text{ constr.}$$

$$x_B' = B^{-1} b' \geq 0 \rightarrow$$ all basic var. must remain non-negative

$$x_B' = B^{-1} b' = B^{-1}(b + \Delta_b) = B^{-1} b + B^{-1} \Delta_b =$$

$$= \begin{pmatrix} 1/2 \\ 15/2 \\ 3 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{8} \\ -\frac{5}{2} & 1 & \frac{1}{8} \\ 0 & 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} \Delta \\ 0 \\ 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 1/2 + 1/2\,\Delta \\ 15/2 - 5/2\,\Delta \\ 3 + 0 \end{pmatrix} \begin{matrix} = x_2' \geq 0 \\ = s_2' \geq 0 \\ = x_1' \geq 0 \end{matrix}$$

$$\begin{cases} 1/2 + 1/2\,\Delta \geq 0 \\ 15/2 - 5/2\,\Delta \geq 0 \end{cases} \rightarrow \text{if } -1 \leq \Delta \leq 3$$

x_B remains optimal

$$z^{*\prime} = c_B^T B^{-1} b' = \underbrace{c_B^T B^{-1} b}_{} + c_B^T B^{-1} \Delta_b =$$

$$= \underbrace{-25.5}_{z^*} + \left(-3/2 \quad 0 \quad -\frac{13}{8}\right) \begin{pmatrix} \Delta \\ 0 \\ 0 \end{pmatrix} =$$

$$= -25.5 - 3/2\,\Delta$$

$\hat{c}_3 \quad \hat{c}_{s_1} \quad \hat{c}_{s_3}$

b) $c_3' = c_3 + \Delta$ $\qquad \hat{c}_N^T = \left(-\frac{9}{8} \;\; -\frac{3}{2} \;\; -\frac{13}{8}\right)$

As x_3 is non-basic variable, the change will affect $\hat{c}_3$

$$\hat{c}_3' = c_B^T B^{-1} A_3 - c_3' = \underbrace{c_B^T B^{-1} A_3 - c_3}_{} - \Delta =$$

$$= \hat{c}_3 - \Delta \leq 0 \quad \text{To keep } x_B \text{ optimal}$$

$$\downarrow$$

$$-\frac{9}{8} - \Delta \leq 0$$

$$\downarrow$$

$$\Delta \geq -\frac{9}{8}$$

c) $c_1' = c_1 + \Delta \rightarrow x_1$ is basic $\rightarrow$ the change affects all $\hat{c}_N$

$$c_B' = c_B + \Delta_{c_B} = \begin{array}{c} x_2 \\ s_2 \\ x_1 \end{array} \begin{pmatrix} -3 \\ 0 \\ -8 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \Delta \end{pmatrix}$$

$$\hat{c}_N' = c_B'^T B^{-1} N - c_N^T = (c_B + \Delta_{c_B})^T B^{-1} N - c_N^T =$$

$$= \underbrace{c_B^T B^{-1} N - c_N^T}_{} + \Delta_{c_B}^T B^{-1} N =$$

$$= \hat{c}_N^T + \Delta_{c_B}^T B^{-1} N \leq 0$$

To keep x_B optimal $\hat{c}_N' \leq 0$

$$\downarrow$$

$$\left(-\tfrac{9}{8} \quad -\tfrac{3}{2} \quad -\tfrac{13}{8}\right) + (0, 0, \Delta)\begin{pmatrix} -5/8 & 1/2 & -1/8 \\ 21/8 & -5/2 & 1/8 \\ 1/4 & 0 & 1/4 \end{pmatrix} =$$

$$= \left(-9/8 + 1/4\,\Delta \; , \; -3/2 + 0 \; , \; -13/8 + 1/4\,\Delta\right)$$

$$\downarrow$$

$$\begin{cases} -9/8 + 1/4\,\Delta \leq 0 \\ -\tfrac{13}{8} + 1/4\,\Delta \leq 0 \end{cases} \rightarrow \begin{cases} \Delta \leq \tfrac{9}{2} \\ \Delta \leq \tfrac{13}{2} \end{cases} \rightarrow \Delta \leq \frac{9}{2}$$

d) $a'_{13} = a_{13} + \Delta$ coeff. for x_3 in 1^{st} constr.

$$A'_3 = A_3 + \begin{pmatrix} \Delta \\ 0 \\ 0 \end{pmatrix}$$

x_3 is non-basic $\rightarrow$ only $\hat{c}_3$ will be affected

$$\hat{c}'_3 = c_B^T B^{-1} A'_3 - c_3 = c_B^T B^{-1}\left(A_3 + \Delta_{A_3}\right) - c_3 =$$

$$= \underbrace{c_B^T B^{-1} A_3 - c_3}\, + c_B^T B^{-1} \Delta_{A_3} =$$

$$= \hat{c}_3 + c_B^T B^{-1} \Delta_{A_3} \leq 0$$ $\rightarrow$ To keep x_B optimal

$$\downarrow$$

$$-\frac{9}{8} + \left(-\tfrac{3}{2} \quad 0 \quad -\tfrac{13}{8}\right)\begin{pmatrix} \Delta \\ 0 \\ 0 \end{pmatrix} =$$

$$= -\frac{9}{8} - \frac{3}{2}\Delta \leq 0 \rightarrow \Delta \geq -\frac{3}{4}$$

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