# Tutorial 7

## Question 1

From the final tableau we have

$$\mathbf{x_B} = (x_2,\, s_2,\, x_1)^T, \quad \mathbf{B}^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{8} \\ -\frac{5}{2} & 1 & \frac{1}{8} \\ 0 & 0 & \frac{1}{4} \end{pmatrix},$$

$$\mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1} = \left(-\frac{3}{2},\, 0,\, -\frac{13}{8}\right), \quad \mathbf{B}^{-1}\mathbf{b} = \left(\frac{1}{2},\, \frac{15}{2},\, 3\right)^T.$$

$$\mathbf{N} = (\mathbf{A}_3\ \mathbf{A}_4\ \mathbf{A}_6), \quad \mathbf{B}^{-1}\mathbf{N} = \begin{pmatrix} -\frac{5}{8} & \frac{1}{2} & -\frac{1}{8} \\ \frac{21}{8} & -\frac{5}{2} & \frac{1}{8} \\ \frac{1}{4} & 0 & \frac{1}{4} \end{pmatrix}$$

**1.a** Let the new right hand side be

$$\mathbf{b}' = \mathbf{b} + (\triangle,\, 0,\, 0)^T$$

The value $\triangle$ must keep the corresponding new basis solution $\mathbf{x}'_\mathbf{B} \geq \mathbf{0}$, i.e. it must satisfy the (vector)inequality

$$\begin{aligned} \mathbf{x}'_\mathbf{B} &= \mathbf{B}^{-1}\mathbf{b}' \\ &= \mathbf{B}^{-1}\left(\mathbf{b} + (\triangle,\, 0,\, 0)^T\right) \\ &= \mathbf{B}^{-1}\mathbf{b} + \mathbf{B}^{-1}(\triangle,\, 0,\, 0)^T \\ &= \left(\tfrac{1}{2},\, \tfrac{15}{2},\, 3\right)^T + \left(\tfrac{\triangle}{2},\, -\tfrac{5\triangle}{2},\, 0\right)^T \\ &= \left(\tfrac{1}{2} + \tfrac{\triangle}{2},\, \tfrac{15}{2} - \tfrac{5\triangle}{2},\, 3\right)^T \geq \mathbf{0} \end{aligned}$$

Hence $\quad -1 \leq \triangle \leq 3.$

**1.b** Let the new coefficient of $x_3$ be $c'_3 = c_3 + \triangle$. Denote the corresponding new reduced cost of $x_3$ by $\widehat{c}'_3$. Since $x_3$ is a non-basic variable, the value $\triangle$ must keep $\widehat{c}'_3 \leq 0$, i.e.

$$\begin{aligned} \widehat{c}'_3 &= \mathbf{c_B}\mathbf{B}^{-1}\mathbf{A}_3 - c'_3 \\ &= \mathbf{c_B}\mathbf{B}^{-1}\mathbf{A}_3 - (c_3 + \triangle) \\ &= (\mathbf{c_B}\mathbf{B}^{-1}\mathbf{A}_3 - c_3) - \triangle \\ &= \widehat{c}_3 - \triangle \\ &= -\tfrac{9}{8} - \triangle \leq 0 \end{aligned}$$

Hence $\qquad \triangle \geq -\frac{9}{8}.$

**1.c** Let the new coefficient of $x_1$ be $c_1' = c_1 + \triangle$. Then the new vector of basic coefficients is $\mathbf{c}_\mathbf{B}^T + (0,\,0,\,\triangle)$. Since $x_1$ is a basic variable, the value $\triangle$ must keep the reduced cost of all nonbasic variables non-positive, i.e.

$$\begin{aligned}
\widehat{\mathbf{c}}_\mathbf{N}'^T &= \mathbf{c}_\mathbf{B}'^T\mathbf{B}^{-1}\mathbf{N} - \mathbf{c}_\mathbf{N}^T \\
&= (\mathbf{c}_\mathbf{B}^T + (0,\,0,\,\triangle))\mathbf{B}^{-1}\mathbf{N} - \mathbf{c}_\mathbf{N}^T \\
&= (\mathbf{c}^T\mathbf{B}^{-1}\mathbf{N} - \mathbf{c}_\mathbf{N}^T) + (0,\,0,\,\triangle)\mathbf{B}^{-1}\mathbf{N} \\
&= \widehat{\mathbf{c}}_\mathbf{N} + (0,\,0,\,\triangle)\begin{pmatrix} -\frac{5}{8} & \frac{1}{2} & -\frac{1}{8} \\ \frac{21}{8} & -\frac{5}{2} & \frac{1}{8} \\ \frac{1}{4} & 0 & \frac{1}{4} \end{pmatrix} \\
&= \Big(-\tfrac{9}{8},\,-\tfrac{3}{2},\,-\tfrac{13}{8}\Big) + \Big(\tfrac{\triangle}{4},\,0,\,\tfrac{\triangle}{4}\Big) \\
&= \Big(-\tfrac{9}{8} + \tfrac{\triangle}{4},\,-\tfrac{3}{2},\,-\tfrac{13}{8} + \tfrac{\triangle}{4}\Big) \leq \mathbf{0}
\end{aligned}$$

$$\text{Hence} \qquad \triangle \leq \min\left\{\frac{9}{2},\,\frac{13}{2}\right\} = \frac{9}{2}.$$

**1.d** Let the new coefficient of $x_3$ in the first constraint be $a_{13}' = a_{13} + \triangle$. Then the new column vector corresponding to $x_3$ in the matrix $\mathbf{A}$ is

$$\mathbf{A}_3' = \mathbf{A}_3 + (\triangle,\,0,\,0)^T$$

Since $x_3$ is a non-basic variable, the value $\triangle$ must keep the new reduced cost of $x_3$ non-positive, i.e.

$$\begin{aligned}
\widehat{c}_3' &= \mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{A}_3' - c_3 \\
&= \mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}(\mathbf{A}_3 + (\triangle,\,0,\,0)^T) - c_3 \\
&= (\mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{A}_3 - c_3) + \mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}(\triangle,\,0,\,0)^T) \\
&= -\tfrac{9}{8} + \Big(-\tfrac{3}{2},\,0,\,-\tfrac{13}{8}\Big)(\triangle,\,0,\,0)^T) \\
&= -\tfrac{9}{8} - \tfrac{3\triangle}{2} \leq 0.
\end{aligned}$$

$$\text{Hence} \qquad \triangle \geq -\frac{3}{4}.$$

## Question 2

From the final tableau we have

$$\mathbf{x}_\mathbf{B} = (x_2,\,x_3)^T, \quad \mathbf{B}^{-1} = \begin{pmatrix} 0.2 & -0.1 \\ 0.1 & 0.2 \end{pmatrix},$$

$$\mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1} = (-0.5,\,-1.5), \quad \mathbf{B}^{-1}\mathbf{b} = (2.5,\,2.5)^T.$$

$$\mathbf{N} = (\mathbf{A}_1\;\mathbf{A}_4\;\mathbf{A}_5), \quad \mathbf{B}^{-1}\mathbf{N} = \begin{pmatrix} -0.1 & 0.2 & -0.1 \\ 0.7 & 0.1 & 0.2 \end{pmatrix}$$

**2.a** Let the new coefficient of $x_3$ be $c_3' = c_3 + \triangle$. Then the new vector of basic coefficients is $\mathbf{c}_\mathbf{B}^T + (0,\, \triangle)$. Since $x_3$ is a basic variable, the value $\triangle$ must keep the reduced cost of all nonbasic variables non-positive. We can do this one-by-one, or combine them into an inequality of vectors, i.e.

$$
\begin{aligned}
\widehat{\mathbf{c}}_\mathbf{N}'^T &= \mathbf{c}_\mathbf{B}'^T \mathbf{B}^{-1}\mathbf{N} - \mathbf{c}_\mathbf{N}^T \\
&= (\mathbf{c}_\mathbf{B}^T + (0,\, \triangle))\mathbf{B}^{-1}\mathbf{N} - \mathbf{c}_\mathbf{N}^T \\
&= (\mathbf{c}^T \mathbf{B}^{-1}\mathbf{N} - \mathbf{c}_\mathbf{N}^T) + (0,\, \triangle)\mathbf{B}^{-1}\mathbf{N} \\
&= \widehat{\mathbf{c}}_\mathbf{N} - (0,\, \triangle)\begin{pmatrix} -0.1 & 0.2 & -0.1 \\ 0.7 & 0.1 & 0.2 \end{pmatrix} \\
&= (-2,\, -0.5,\, -1.5) + (0.7\triangle,\, 0.1\triangle,\, 0.2\triangle) \\
&= (0.7\triangle - 2,\, 0.1\triangle - 0.5,\, 0.2\triangle - 0.15) \;\le\; \mathbf{0}
\end{aligned}
$$

Hence $$\triangle \;\le\; \min\left\{\frac{2}{0.7}, \frac{0.5}{0.1}, \frac{1.5}{0.2}\right\} \;=\; \frac{2}{0.7} \;\approx\; 2.857.$$

**2.b** Let the new coefficient of $x_1$ in the first constraint be $a_{11}' = a_{11} + \triangle$. Then the new column vector corresponding to $x_3$ in the matrix $\mathbf{A}$ is

$$\mathbf{A}_1' = \mathbf{A}_1 + (\triangle,\, 0)^T$$

Since $x_1$ is a non-basic variable, the value $\triangle$ must keep the new reduced cost of $x_1$ non-positive, i.e.

$$
\begin{aligned}
\widehat{c}_1' &= \mathbf{c}_\mathbf{B}\mathbf{B}^{-1}\mathbf{A}_1' - c_1 \\
&= \mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}(\mathbf{A}_1 + (\triangle,\, 0)^T) - c_1 \\
&= (\mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{A}_1 - c_1) + \mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}(\triangle,\, 0)^T) \\
&= -2 + (-0.5,\, -1.5)(\triangle,\, 0)^T) \\
&= -2 - 0.5\triangle \;\le\; 0.
\end{aligned}
$$

Hence $$\triangle \;\ge\; -\frac{2}{0.5} \;=\; -4.$$

**2.c i.** Rewrite the new hand side in the form

$$\mathbf{b}' = \mathbf{b} + (0,\, \triangle)^T$$

The value $\triangle$ must keep the corresponding new basis solution $\mathbf{x}_\mathbf{B}' \ge \mathbf{0}$, i.e. it must satisfy the (vector)inequality

$$
\begin{aligned}
\mathbf{x}_\mathbf{B}' &= \mathbf{B}^{-1}\mathbf{b}' \\
&= \mathbf{B}^{-1}\big(\mathbf{b} + (0,\, \triangle)^T\big) \\
&= \mathbf{B}^{-1}\mathbf{b} + \mathbf{B}^{-1}(0,\, \triangle)^T \\
&= (2.5,\, 2.5)^T + \begin{pmatrix} 0.2 & -0.1 \\ 0.1 & 0.2 \end{pmatrix}(0,\, \triangle)^T \\
&= (2.5,\, 2.5)^T + (-0.1\triangle,\, 0.2\triangle)^T \\
&= (2.5 - 0.1\triangle,\, 2.5 + 0.2\triangle)^T \;\ge\; \mathbf{0}
\end{aligned}
$$

Hence $$-\frac{2.5}{0.2} \le \triangle \le \frac{2.5}{0.1} \quad \Leftrightarrow \quad -12.5 \le \triangle \le 25.$$

**ii.** The new objective value is

$$\mathbf{c}_{\mathbf{B}}^T \mathbf{B}^{-1} \mathbf{b}' \;=\; (-0.5,\, -1.5) \begin{pmatrix} 15 \\ 5 + \triangle \end{pmatrix} \;=\; -15 - 1.5\triangle$$

**2.d** The optimal objective value is -15.5.

**2.e** The optimal objective value is -16.5.

**2.f** The vector $\mathbf{c}_{\mathbf{B}}^T \mathbf{B}^{-1} = (-0.5, -1.5)$ is called the vector of shadow prices (or shadow costs). This vector gives the change in objective value if the right hand side of the corresponding row is increased by one unit. In part (**2 d**), the right hand side of the first row increased by one unit, so the objective value now is $-15 + (-0.5) = -15.5$. In part (**2.e**), the right hand side of the second row increased by one unit, so the objective value now is $-15 + (-1.5) = -16.5$.