37242 Introduction to Optimisation

Tutorial 9

1. (slightly modified from Nash and Sofer, p.434)

   Consider the problem

$$\begin{array}{rl} \min & f(\mathbf{x}) = x_1^2 + x_1^2 x_3^2 + 2x_1x_2 + x_2^4 + 8x_2 \\ \text{s.t.} & 2x_1 + 5x_2 + x_3 = 3 \end{array}$$

   (a) Determine which of the following points are stationary points:

$$\text{(i)} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \text{(ii)} \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}, \text{ and (iii)} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

   (b) Determine whether each stationary point is a local minimiser.

2. Consider the problem

$$\begin{array}{rrcrcrcr} \min & x_1^2 + 3x_1x_2 + 9x_2^2 + x_3^2 & & & & & & \\ \text{s.t.} & x_1 & - & x_2 & + & x_3 & = & 4 \\ & 2x_1 & + & x_2 & + & 5x_3 & = & 8 \end{array}$$

   (a) Write down the Lagrangian function for this problem.

   (b) Use the Lagrangian to confirm that the solution $(x_1^*, x_2^*, x_3^*) = (2.6, -0.7, 0.7)$ is a stationary point for the above constrained optimisation problem (you will need to find optimal values for the Lagrange multipliers μ_1 and μ_2).

3. Consider the problem

$$\begin{array}{rl} \min & f(x_1, x_2) = x_1^2 + x_1x_2 + x_2^2 + 4x_1. \\ \text{s.t.} & x_1 + 3x_2 = 1 \end{array}$$

   (a) Write down the Lagrangian function for this problem.

   (b) Use the Lagrangian to find all stationary points for this problem.

4. (slightly modified from Nash and Sofer, p.437)

Consider the problem

$$\begin{array}{ll} \min & f(x_1, x_2, x_3) = 3x_1^2 - \frac{1}{2}x_2^2 - \frac{1}{2}x_3^2 + x_1x_2 - x_1x_3 + 2x_2x_3 \\ \text{s.t.} & 2x_1 - x_2 + x_3 = 2. \end{array}$$

(a) Find a stationary point for this problem using the Lagrangian.

(b) Show that this is a local minimum.

Lagrangian function – equality constraints

Theorem 3. - constrained

➢ If (x^*, Λ^*) is a stationary point to $L(x, \Lambda)$:

1. • $\frac{\partial L(x,\Lambda)}{\partial \lambda_i} = 0, i = 1..m$ $\rightarrow$ Feasibility of x^*

2. • $\frac{\partial L(x,\Lambda)}{\partial x_j} = 0, j = 1..n$ $\rightarrow$ $\underbrace{\nabla f(x^*) = A^T \Lambda^*}$ $\rightarrow$ $Z^T \nabla f(x^*) =$

$= Z^T A^T \Lambda^* = 0$

3. *Each* $g_i(x)$ is linear *And* *f(x)* is a convex function,

then x^* is a local minimum of *f(x) on* $\{\underbrace{g(x) = b}\}$

if $f(x)$ is not convex, then

check if $Z^T \nabla^2 f(x^*) Z$ is

pos. - def

4. (slightly modified from Nash and Sofer, p.437)

Consider the problem

$$\begin{array}{ll} \min & f(x_1,x_2,x_3) = 3x_1^2 - \frac{1}{2}x_2^2 - \frac{1}{2}x_3^2 + x_1x_2 - x_1x_3 + 2x_2x_3 \\ \text{s.t.} & 2x_1 - x_2 + x_3 = 2. \end{array}$$

(a) Find a stationary point for this problem using the Lagrangian.

(b) Show that this is a local minimum.

$$L(x,\lambda) = \underbrace{3x_1^2 - \frac{1}{2}x_2^2 - \frac{1}{2}x_3^2 + x_1x_2 - x_1x_3 + 2x_2x_3}_{f(x)} + \lambda(2 - 2x_1 + x_2 - x_3)$$

Find where $\nabla L(x,\lambda) = 0$:

$$\begin{array}{ll} x_1 & \\ x_2 & \\ x_3 & \\ \lambda & \end{array} \begin{cases} 6x_1 + x_2 - x_3 - 2\lambda = 0 \\ -x_2 + x_1 + 2x_3 + \lambda = 0. \\ -x_3 - x_1 + 2x_2 - \lambda = 0. \\ 2 - 2x_1 + x_2 - x_3 = 0 \end{cases}$$

Use $(A|b) \sim (I|A^{-1}b)$

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & \lambda & b \end{array} \\ \left(\begin{array}{cccc|c} 6 & 1 & -1 & -2 & 0 \\ 1 & -1 & 2 & 1 & 0 \\ -1 & 2 & -1 & -1 & 0 \\ 2 & -1 & 1 & 0 & 2 \end{array}\right) \end{array} \quad R_1 \leftrightarrow R_2 \quad \sim$$

$$\left(\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 0 \\ 0 & 7 & -13 & -8 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 2 \end{array}\right) \quad \begin{array}{l} R_2' = R_2 - 6R_1 \\ R_3' = R_3 + R_1 \\ R_4' = R_4 - 2R_1 \end{array} \quad R_3 \longleftrightarrow R_2 \quad \sim$$

$$\left(\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -20 & -8 & 0 \\ 0 & 0 & -4 & -2 & 2 \end{array}\right) \quad \begin{array}{l} R_3' = R_3 - 7R_2 \\ R_4' = R_4 - R_2 \end{array} \quad \sim$$

$$\left(\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{2}{5} & 0 \\ 0 & 0 & 0 & -\frac{2}{5} & 2 \end{array}\right) \quad \begin{array}{l} R_3' = R_3/(-20) \\ R_4' = R_4 + 4R_3' \end{array} \quad \sim$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -5 \end{array}\right) \quad \begin{array}{l} R_1' = R_1 + R_2' - 2R_3' - R_4' \\ R_2' = R_2 - R_3' \\ R_3' = R_3 + R_4 \\ R_4' = R_4 / -\frac{2}{5} \end{array}$$

The only stationary point of $L(x, \lambda)$ is $(x^*, \lambda^*)^T = (\underbrace{-1, -2, 2}_{x^*}, \underbrace{-5}_{\lambda})$

is $f(x)$ convex? Need $\nabla^2 f$

$$\nabla f(x) = \begin{pmatrix} 6x_1 + x_2 - x_3 \\ -x_2 + x_1 + 2x_3 \\ -x_3 + 2x_2 - x_1 \end{pmatrix}$$

$$\nabla^2 f(x) = \begin{pmatrix} 6 & 1 & -1 \\ 1 & -1 & 2 \\ -1 & 2 & -1 \end{pmatrix}$$

Check principal minors:

1st p.m. $6, -1, -1$ – not convex

· not concave

$\downarrow$

by Th 1 (unconstrained)

$f(x)$ is convex iff all p.m. of $\nabla^2 f$ are non-negative

$\downarrow$

$f(x)$ is not convex

$\downarrow$

check whether $Z^T \nabla^2 f(x^*) Z$ is pos.-def.

$\downarrow$

need Z

$$Z = \begin{pmatrix} -B^{-1}N \\ I \end{pmatrix}$$

$$2x_1 - x_2 + x_3 = 2$$

$$A = (2, -1, 1)$$

$$= (1/2, -1/2) \quad x_B = x_1 \; ; \; x_N = (x_2 \; x_3)$$

$$B = 2 \; ; \; B^{-1} = \frac{1}{2} \; ; \; N = (-1, 1)$$

$$Z = \begin{pmatrix} 1/2 & -1/2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad -B^{-1}N = -1/2(-1, 1)$$

$$Z^T \nabla^2 f(x) Z = \begin{pmatrix} 1/2 & 1 & 0 \\ -1/2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 6 & 1 & -1 \\ 1 & -1 & 2 \\ -1 & 2 & -1 \end{pmatrix} \begin{pmatrix} 1/2 & -1/2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -1/2 & 3/2 \\ -4 & 3/2 & -1/2 \end{pmatrix} \begin{pmatrix} 1/2 & -1/2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3/2 & -1/2 \\ -1/2 & 3/2 \end{pmatrix}$$

Need to show that the reduced Hessian is pos. - def :

$$\det \begin{pmatrix} 3/2 - \alpha & -1/2 \\ -1/2 & 3/2 - \alpha \end{pmatrix} = 0 \rightarrow (3/2 - \alpha)^2 - 1/4 = 0$$

$$3/2 - \alpha = \frac{1}{2} \qquad 3/2 - \alpha = -1/2$$

$$\alpha_1 = 1 > 0 \qquad \alpha_2 = 2 > 0$$

by Th. 4 (unconstr.)

reduced Hessian is pos.-def.

$\downarrow$

by Th. 3 (constrained)

$x^* = \begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix}$ is local min for constr. NLP.

OR Find the min using Reduced gradient and reduced Hessian (Th. 2 - constr)

1. Find where $Z^T \nabla f = 0$

$$\begin{pmatrix} 1/2 & 1 & 0 \\ -1/2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 6x_1 + x_2 - x_3 \\ -x_2 + x_1 + 2x_3 \\ -x_3 + 2x_2 - x_1 \end{pmatrix} =$$

$$= \begin{cases} 3x_1 + \frac{x_2}{2} - \frac{1}{2}x_3 + x_1 - x_2 + 2x_3 = 0 \\ -3x_1 - \frac{1}{2}x_2 + \frac{1}{2}x_3 - x_3 + 2x_2 - x_1 = 0. \end{cases}$$

$$\begin{cases} 4x_1 - \frac{1}{2}x_2 + \frac{3}{2}x_3 = 0. \\ -4x_1 + \frac{3}{2}x_2 - \frac{1}{2}x_3 = 0 \\ 2x_1 - x_2 + x_3 = 2 \end{cases}$$

$$\left(\begin{array}{ccc|c} 4 & -\frac{1}{2} & \frac{3}{2} & 0 \\ -4 & \frac{3}{2} & -\frac{1}{2} & 0 \\ 2 & -1 & 1 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -\frac{1}{8} & \frac{3}{8} & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -\frac{3}{4} & \frac{1}{4} & 2 \end{array}\right) \sim$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right) \rightarrow x^* = \begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix}$$

x^* is feas. and $z^T \nabla f(x^*) = 0$

as $z^T \nabla^2 f(x^*) z$ is pos. def $\rightarrow$ x^* is local min of constr. NLP